DOUBLE CONNECTED SPACES

Ali Saad Kandil\textsuperscript{1} <dr.ali-kandil@yahoo.com>
Osama Abd El-Hamed El-Tantawy\textsuperscript{2} <drosamat@yahoo.com>
Sobhy Ahmed Ali El-Sheikh\textsuperscript{3} <sobhyesheikh@yahoo.com>
Salama Hussien Ali Shaliel\textsuperscript{3,*} <slamma-elarabi@yahoo.com>

\textsuperscript{1}Mathematics Department, Faculty of Science, Helwan University, Helwan, Egypt.
\textsuperscript{2}Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt.
\textsuperscript{3}Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt.

Abstract — In this paper, we introduce new types of double connected topological spaces. The first one depends on the separated double sets and the other one depends on the quasi-coincident separated double sets. The properties and the relation between them have investigated. Also, we defined and study the component of each type and the properties of these types of component have obtained.

Keywords — Separated double sets, q-separated double sets, double connected, q-double connected, DC\textsubscript{1}-connected, strongly double connected, strongly q-double connected, double components, q-double components and q-hyperconnected in double topological spaces.

1 Introduction

Atanassov \textsuperscript{[1, 2, 3, 4]} introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker \textsuperscript{[5]} generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Coker \textsuperscript{[7, 6]}.

In 2005, the suggestion of J. G. Garcia et al. \textsuperscript{[8]} that double set is a more appropriate name than flou (intuitionistic) set, and double topology for the flou (intuitionistic) topology. In 2007, Kandil et al. \textsuperscript{[10]} proved the 1 – 1 correspondence mapping \( f \) between the set of all double sets and the set of all intuitionistic sets defined as: \( f(A_1, A_2) = (A_1, A_2^c) \), \( A_2^c \) is the complement of \( A_2 \). Kandil et al. \textsuperscript{[9, 10]} introduced the concept of double sets, (D-set, for short), double topological spaces.
In this paper, we define \((q\text{-})\)separated double sets, \((q)\)double connected, \(DC_1\)-connected, strongly double connected, \((q)\)double components and quasi-hyperconnected in double topological spaces. Moreover, we give some related results to these notions.

## 2 Preliminary

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [9, 10].

**Definition 2.1.** [10] Let \(X\) be a non-empty set.

1. A D-set \(A\) is an ordered pair \((A_1, A_2) \in P(X) \times P(X)\) such that \(A_1 \subseteq A_2\).
2. \(D(X) = \{(A_1, A_2) \in P(X) \times P(X), A_1 \subseteq A_2\}\) is the family of all D-sets on \(X\).
3. Let \(\eta_1, \eta_2 \subseteq P(X)\). The product of \(\eta_1\) and \(\eta_2\), denoted by \(\eta_1 \times \eta_2\), defined by:
   \[
   \eta_1 \times \eta_2 = \{(A_1, A_2) : A_1 \in \eta_1, A_1 \subseteq A_2\}.
   \]
4. The D-set \(X = (X, X)\) is called the universal D-set.
5. The D-set \(\emptyset = (\emptyset, \emptyset)\) is called the empty D-set.

**Definition 2.2.** [10] Let \(A = (A_1, A_2), B = (B_1, B_2)\) and \(C = (C_1, C_2) \in D(X)\).

1. \(A = B \iff A_1 = B_1, A_2 = B_2\).
2. \(A \subseteq B \iff A_1 \subseteq B_1, A_2 \subseteq B_2\).
3. \(A \cup B = (A_1 \cup B_1, A_2 \cup B_2)\).
4. \(A \cap B = (A_1 \cap B_1, A_2 \cap B_2)\).
5. \(A^c = (A_2^c, A_1^c)\), where \(A^c\) is the complement of \(A\).
6. \(A \setminus B = (A_1 \setminus B_2, A_2 \setminus B_1)\).
7. \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\).
8. Let \(x \in X\). Then, the D-sets \(\bar{A}_1 = (\{x\}, \{x\})\) and \(\bar{A}_2 = (\emptyset, \{x\})\) are said to be D-points in \(X\). The family of all D-points, denoted by \(DP(X)\), i.e., \(DP(X) = \{\bar{A}_t : x \in X, t \in \{\frac{1}{2}, 1\}\}\).
9. \(x_1 \in A \iff x \in A_1\) and \(x_2 \in A \iff x \in A_2\).

**Definition 2.3.** [9] Two D-sets \(A\) and \(B\) are said to be a quasi-coincident, denoted by \(A \# B\), if \(A_1 \cap B_2 \neq \emptyset\) or \(A_2 \cap B_1 \neq \emptyset\). \(A\) is not quasi-coincident with \(B\), denoted by \(A \not\# B\), if \(A_1 \cap B_2 = \emptyset\) and \(A_2 \cap B_1 = \emptyset\).
Theorem 2.4. [9] Let $A, B, C \in D(X)$ and $x \in DP(X)$. Then,

1. $A \not\preceq B \iff A \subseteq B^c$.

2. $A \not\preceq B, C \subseteq B \Rightarrow A \not\preceq C$.

Definition 2.5. [10] Let $X$ be a non-empty set. The family $\eta$ of D-sets in $X$ is called a double topology on $X$ if it satisfies the following axioms:

1. $\emptyset, X \in \eta$,

2. If $A, B \in \eta$, then $A \cap B \in \eta$,

3. If $\{A_s : s \in S\} \subseteq \eta$, then $\bigcup_{s \in S} A_s \in \eta$.

The pair $(X, \eta)$ is called a DTS. Each element of $\eta$ is called an open D-set in $X$. The complement of open D-set is called closed D-set.

Definition 2.6. [10] Let $(X, \eta)$ be a DTS and $A \in D(X)$. The double closure of $A$, denoted by $cl_\eta(A)$ or $\overline{A}$, defined by: $cl_\eta(A) = \bigcap \{B : B \in \eta \text{ and } A \subseteq B\}$.

Theorem 2.7. [10] Let $(X, \eta)$ be a DTS and let $A, B, C \in D(X)$. Then,

1. $cl_\eta(A)$ is the smallest closed D-set containing $A$.

2. $cl_\eta(A \cap B) \subseteq cl_\eta(A) \cap cl_\eta(B)$.

3. $A \not\preceq C \iff cl_\eta(A) \not\preceq cl_\eta(C), C \in \eta$.

Definition 2.8. [10] Let $X$ be a non-empty set. The family $\tau$ of D-sets in $X$ is called a stratified double topology on $X$ if it satisfies the following axioms:

1. $\emptyset, X \in \tau$ and $(\emptyset, X) \in \tau$,

2. If $A, B \in \tau$, then $A \cap B \in \tau$,

3. If $\{A_s : s \in S\} \subseteq \tau$, then $\bigcup_{s \in S} A_s \in \tau$.

The pair $(X, \tau)$ is called a stratified DTS.

Definition 2.9. [10] Let $X$ be a non-empty set.

1. $I(X) = \{\emptyset, X\}$ is a DTS, which is called indiscrete DTS.

2. $i(X) = \{(\emptyset, X), (\emptyset, X)\}$ is a DTS, which is called indiscrete stratified DTS, $i(X)$ is the indiscrete topology on $X$.

3. $D(X) = P(X) \times P(X)$ is a DTS, which is called discrete DTS.

Theorem 2.10. [10] Let $\eta$ be a double topology on $X$. Then, the following collections are ordinary topologies on $X$:

1. $\pi_1 = \{A_1 : A \in \eta\}$. 
2. $\pi_2 = \{A_2 : A \in \eta\}$.

**Definition 2.11.** [10] Let $(X, \eta)$ be a DTS and $Y$ be a non-empty subset of $X$. Then, $\eta_Y = \{A \cap Y : A \in \eta \text{ and } Y = (Y, Y)\}$ is a double topology on $Y$. The DTS $(Y, \eta_Y)$ is called a double topological subspace of $(X, \eta)$ (DT-subspace, for short).

**Definition 2.12.** [10] Let $(X, \eta)$ be a DTS, $F \in D(X)$ and $Y$ be a non-empty subset of $X$. Then, the D-subset over $Y$, denoted by $F^Y$, defined by: $F^Y = F \cap Y$.

**Definition 2.13.** [10] Consider two ordinary sets $X$ and $Y$. Let $f$ be a mapping from $X$ into $Y$. The image of a D-set $A$ in $D(X)$ defined by: $f(A) = (f(A_1), f(A_2))$. Also the inverse image of a D-set $B$ in $D(Y)$ defined by: $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.

**Definition 2.14.** [10] Let $f : X \to Y$ be a mapping and let $(X, \eta)$ and $(Y, \eta^*)$ be DTS. Then, $f$ is called a D-continuous if $f^{-1}(B) \in \eta$, whenever $B \in \eta^*$.

**Theorem 2.15.** [10] Let $(X, \eta)$ and $(Y, \eta^*)$ be two DTS and let $f : X \to Y$ be a mapping, $A \in D(X)$ and $B \in D(Y)$. Then, the following conditions are equivalent:

1. $f$ is a D-continuous,
2. $f^{-1}(B) \in \eta^c$, $\forall B \in \eta^*$,
3. $f(cl_\eta(A)) \subseteq cl_{\eta^*}(f(A))$, $\forall A \in D(X)$,
4. $cl_\eta(f^{-1}(B)) \subseteq f^{-1}(cl_{\eta^*}(B))$, $\forall B \in D(Y)$,
5. $f^{-1}(int_{\eta^*}(B)) \subseteq int_\eta(f^{-1}(B))$, $\forall B \in D(Y)$.

**Definition 2.16.** [10] Let $(X, \eta)$ and $(Y, \eta^*)$ be two DTS and let $f : X \to Y$ be a mapping and $A \in D(X)$.

1. $f$ is called D-open if $f(A) \in \eta^*$, $\forall A \in \eta$.
2. $f$ is called D-closed if $f(A) \in \eta^c$, $\forall A \in \eta^c$.

**Theorem 2.17.** [10] Let $(X, \eta)$ and $(Y, \eta^*)$ be two DTS and let $f : X \to Y$ be a mapping and $A \in D(X)$. $f$ is D-closed iff $cl_{\eta^*}(f(A)) \subseteq f(cl_\eta(A))$, $\forall A \in D(X)$.

**Definition 2.18.** [9] Let $(X, \eta)$ be a DTS and let $A \in D(X)$. $A$ is said to be:

1. D-dense if $cl_\eta(A) = X$.
2. D-nowhere dense if $int_\eta(cl_\eta(A)) = \emptyset$. 
3 Connectedness in \( DTS \)

**Definition 3.1.** Let \((X, \eta)\) be a \(DTS\) and let \( A, B \in D(X) \).

1. \( A, B \) are said to be separated double sets (separated D-sets, for short) if 
   \[ cl_\eta(A) \cap B = \emptyset \text{ and } A \cap cl_\eta(B) = \emptyset. \]

2. \( A, B \) are said to be quasi-coincident separated double sets (q-separated D-sets, for short) if 
   \[ cl_\eta(A) \not\subseteq B \text{ and } A \not\subseteq cl_\eta(B). \]

**Proposition 3.2.** Let \((X, \eta)\) be a \(DTS\) and let \( A, B \in D(X) \). Then, if \( A, B \) are separated D-sets, then \( A \cap B = \emptyset. \)

**Proof.** Suppose \( A, B \) are separated D-sets, then \( cl_\eta(A) \cap B = \emptyset \) and \( A \cap cl_\eta(B) = \emptyset. \)

But \( A \subseteq cl_\eta(A), \text{ then } A \cap B = \emptyset. \) Hence, the result.

The following example shows that the converse of Proposition 3.2 is not true in general.

**Example 3.3.** Let \( X = \{a, b, c, d\}, \eta = \{\emptyset, X, (\{a\}, \{a, b\}), (\emptyset, \{a, b\}), (\emptyset, \emptyset)\} \) and \( \eta' = \{\emptyset, X, (\{a\}, \{a, b\}), (\emptyset, \{a, b\}), (\emptyset, \emptyset)\}. \) Then, \((X, \eta)\) is a \(DTS\). Now, \( (\{a\}, \{a, b\}) \cap (\{c\}, \{c, d\}) = \emptyset, \) but \( X = cl_\eta(\{a\}, \{a, b\}) \cap (\{c\}, \{c, d\}) = (\{c\}, \{c, d\}) \neq \emptyset. \)

**Proposition 3.4.** Let \((X, \eta)\) be a \(DTS\) and let \( A, B \in D(X) \). Then, if \( A, B \) are q-separated D-sets, then \( A \not\subseteq B. \)

**Proof.** Suppose \( A, B \) are q-separated D-sets, then \( cl_\eta(A) \not\subseteq B \) and \( A \not\subseteq cl_\eta(B) \). But \( A \subseteq cl_\eta(A), \text{ then } A \not\subseteq B. \) Hence, the result.

The following example shows that the converse of Proposition 3.4 is not true in general.

**Example 3.5.** Let \( X = \{a, b, c\} \) and \( \eta = \{\emptyset, X, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \emptyset)\}. \) Then, \( \exists(\{a\}, \{a\}), (\{b\}, \{b\}) \in D(X) \) such that \((\{a\}, \{a\}) \not\subseteq (\{b\}, \{b\}) \). But, \( cl_\eta(\{a\}, \{a\}) = (\{a, c\}, X) \not\subseteq (\{b\}, \{b\}) \) and \( cl_\eta(\{b\}, \{b\}) = (\{b, c\}, X) \not\subseteq (\{a\}, \{a\}). \)

**Proposition 3.6.** Let \((X, \eta)\) be a \(DTS\) and let \( A, B \in D(X) \). Then, if \( A \cap B = \emptyset, \text{ then } A \not\subseteq B. \)

**Proof.** Suppose that \((A_1, A_2) = A \cap B = (B_1, B_2) = \emptyset, \text{ then } A_1 \cap B_1 = \emptyset \) and \( A_2 \cap B_2 = \emptyset, \) but \( A_1 \subseteq A_2, B_1 \subseteq B_2, \) then \( A_1 \cap B_2 = \emptyset \) and \( A_2 \cap B_1 = \emptyset. \) Therefore, \( A \not\subseteq B. \)

The following example shows that the converse of Proposition 3.6 is not true in general.

**Example 3.7.** Let \( X = \{a, b\} \) and \( \eta = \{\emptyset, X, (\{a\}, \{a\}), (\emptyset, \{b\}), (\{a\}, X)\}. \) Then, \( (\emptyset, \{b\}) \not\subseteq (\{a\}, X), \) but \( (\emptyset, \{b\}) \cap (\{a\}, X) = (\emptyset, \{b\}) \neq \emptyset. \)

**Proposition 3.8.** Let \((X, \eta)\) be a \(DTS\) and let \( A, B \in D(X) \). Then, if \( A, B \) are separated D-sets, then \( A, B \) are q-separated D-sets.
Proof. Straightforward.

The following example shows that the converse of Proposition 3.8 is not true in general.

**Example 3.9.** In Example 3.7, we see that: \((\emptyset, \{b\}) \notin cl_\eta(\{a\}, X) = (\{a\}, X)\) and \(cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \notin (\{a\}, X)\), but \((\emptyset, \{b\}) = cl_\eta(\emptyset, \{b\}) \cap (\{a\}, X) = (\emptyset, \{b\}) \neq \emptyset\).

**Remark 3.10.** \(A \cap B = \emptyset \Leftrightarrow cl_\eta(A) \notin B\) and \(A \notin cl_\eta(B)\).

**Example 3.11.**
1. In Example 3.3, we see that: \((\{a\}, \{a, b\}) \cap (\{c\}, \{c, d\}) = \emptyset\), but \(\overline{X} = cl_\eta(\{a\}, \{a, b\}) \notin (\{c\}, \{c, d\})\).
2. In Example 3.7, we see that: \((\emptyset, \{b\}) \notin cl_\eta(\{a\}, X) = (\{a\}, X)\) and \(cl_\eta(\emptyset, \{b\}) = (\emptyset, \{b\}) \notin (\{a\}, X)\), but \((\emptyset, \{b\}) \cap (\{a\}, X) = (\emptyset, \{b\}) \neq \emptyset\).

**Theorem 3.12.** Let \((X, \eta)\) be a DT S and let \(A, B, C, D \in D(X)\) such that \(C \subseteq A\) and \(D \subseteq B\). Then, if \(A, B\) are separated D-sets, then \(C, D\) are separated D-sets.

**Proof.** Suppose \(A, B\) are separated D-sets, then \(cl_\eta(A) \cap B = \emptyset\) and \(A \cap cl_\eta(B) = \emptyset\). Since \(C \subseteq A\) and \(D \subseteq B\), then \(cl_\eta(C) \subseteq cl_\eta(A)\) and \(cl_\eta(D) \subseteq cl_\eta(B)\). Implies, \(C \subseteq A\) and \(D \subseteq B\). Then, if \(A, B\) are separated D-sets, then \(C, D\) are separated D-sets.

**Theorem 3.13.** Let \((X, \eta)\) be a DT S and let \(A, B, C, D \in D(X)\) such that \(C \subseteq A\) and \(D \subseteq B\). Then, if \(A, B\) are q-separated D-sets, then \(C, D\) are q-separated D-sets.

**Proof.** Suppose \(A, B\) are q-separated D-sets, then \(cl_\eta(A) \notin B\) and \(A \notin cl_\eta(B)\). Since \(C \subseteq A\) and \(D \subseteq B\), then \(cl_\eta(C) \subseteq cl_\eta(A)\) and \(cl_\eta(D) \subseteq cl_\eta(B)\). Implies, \(C \notin cl_\eta(B)\) and \(D \notin cl_\eta(C)\) [by Theorem 2.4]. Hence, \(C, D\) are q-separated D-sets.

**Definition 3.14.** Let \((X, \eta)\) be a DT S, and let \(E\) be a nonempty subset of \(X\).

1. If there exist two non-empty separated D-sets \(A, B \in D(X)\) such that \(A \cup B = E\), then the D-sets \(A\) and \(B\) form a D-separation of \(E\) and it is said to be double disconnected set (D-disconnected set, for short). Otherwise, \(E\) is said to be double connected set (D-connected set, for short).
2. If there exist two non-empty q-separated D-sets \(A, B \in D(X)\) such that \(A \cup B = E\), then the D-sets \(A\) and \(B\) form a qD-separation of \(E\) and it is said to be quasi-coincident double disconnected set (qD-disconnected set, for short). Otherwise, \(E\) is said to be quasi-coincident double connected set (qD-connected set, for short).

**Remark 3.15.** The D-point \(x_1\) in any DT S \((X, \eta)\) is a qD-connected set, provided that \(x_1 \notin cl_\eta(x_1)\).

**Example 3.16.** Let \(X = \{a, b\}\), \(\eta = \{\emptyset, X, (\emptyset, \{a\}), (\{b\}, X)\}\) and \(\eta^c = \{\emptyset, X, (\emptyset, \{a\}), (\{b\}, X)\}\). Then,

1. \((\emptyset, \{a\}) = cl(\emptyset, \{a\}) \notin (\emptyset, \{a\})\) and \((\emptyset, \{a\}) \cup (\emptyset, \{a\}) = (\emptyset, \{a\})\). Therefore, \((\emptyset, \{a\})\) is not qD-connected set.
2. \((\{b\}, X) = cl(\emptyset, \{b\}) \ q (\emptyset, \{b\}) \ q (\emptyset, \{b\}) = (\emptyset, \{b\})\). Therefore, 
\((\emptyset, \{b\})\) is a \(qD\)–connected set.

**Definition 3.17.** Let \((X, \eta)\) be a \(DTS\).

1. If there exist two non-empty separated \(D\)-sets \(A, B \in D(X)\) such that \(A \cup B = X\), then \(A\) and \(B\) are said to be double division (\(D\)-division, for short) for \(DTS\) \((X, \eta)\). \((X, \eta)\) is said to be double disconnected space (\(D\)-disconnected space, for short), if \((X, \eta)\) has a \(D\)-division. Otherwise, \((X, \eta)\) is said to be double connected space (\(D\)-connected space, for short).

2. If there exist two non-empty \(q\)-separated \(D\)-sets \(A, B \in D(X)\) such that \(A \cup B = X\), then \(A\) and \(B\) are said to be a double quasi division (\(qD\)-division, for short) for \(DTS\) \((X, \eta)\). \((X, \eta)\) is said to be quasi-coincident double disconnected space (\(qD\)-disconnected space, for short), if \((X, \eta)\) has a \(qD\)-division. Otherwise, \((X, \eta)\) is said to be quasi-coincident double connected space (\(qD\)-connected space, for short).

**Corollary 3.18.** 1. Each indiscrete (stratified) \(DTS\) is \(D\)-connected (\(qD\)-connected).

2. Each discrete \(DTS\) is \(D\)-disconnected (\(qD\)-disconnected).

**Proof.** 1. It is obvious.

2. Suppose that \((X, \eta)\) is a discrete \(DTS\), then \(x_1 \in DP(X)\). Implies, \(X = x_1 \cup x_1^c\), \(x_1 = cl_\eta(x_1) \cap x_1^c = \emptyset\) and \(x_1^c = cl_\eta(x_1^c) \cap x_1 = \emptyset\). Therefore, \((X, \eta)\) is a \(D\)-disconnected. Similarly, \((X, \eta)\) is a \(qD\)-disconnected.

**Theorem 3.19.** Let \((X, \eta)\) be a \(DTS\). Then, the following are equivalent:

1. \((X, \eta)\) has a \(D\)-division,

2. There exist two disjoint closed \(D\)-sets \(A\) and \(B\) such that \(A \cup B = X\),

3. There exist two disjoint open \(D\)-sets \(A\) and \(B\) such that \(A \cup B = X\).

**Proof.** (1 \(\rightarrow\) 2) Suppose that \((X, \eta)\) has a \(D\)-division \(A\) and \(B\), then \(A \cup B = X\) and \(cl_\eta(A) \cap B = \emptyset\). Implies, \(cl_\eta(A) \subseteq B^c = X \setminus B \subseteq A\), but \(A \subseteq cl_\eta(A)\), so that \(A = cl_\eta(A)\). Therefore, \(A\) is a closed \(D\)-set. Similarly, we can see that \(B\) is also a closed \(D\)-set. Since \(A = cl_\eta(A)\), \(cl_\eta(A) \cap B = \emptyset\), then \(A \cap B = \emptyset\). Hence, the result.

(2 \(\rightarrow\) 3) Suppose that \((X, \eta)\) has a \(D\)-division \(A\) and \(B\) such that \(A\) and \(B\) are closed \(D\)-sets, then \(A^c\) and \(B^c\) are open \(D\)-sets, \(A = B^c\) and \(B = A^c\). Therefore, \(A^c \cup B^c = X\) and \(A^c \cap B^c = \emptyset\). Hence, the result.

(3 \(\rightarrow\) 1) Since \(X = A \cup B\) such that \(A \cap B = \emptyset\) and \(A, B\) are open \(D\)-sets, then \(A^c\) and \(B^c\) are closed \(D\)-sets, \(A = B^c\) and \(B = A^c\). This implies that, \(A = cl_\eta(A)\). Therefore, \(cl_\eta(A) \cap B = \emptyset\). Similarly, we have \(A \cap cl_\eta(B) = \emptyset\). Hence, \((X, \eta)\) has a \(D\)-division.

**Theorem 3.20.** Let \((X, \eta)\) be a \(DTS\). Then, the following are equivalent:

1. \((X, \eta)\) has a \(qD\)-division,
2. There exist two non quasi-coincident closed D-sets \( A \) and \( B \) such that \( A \cup B = X \).

3. There exist two non quasi-coincident open D-sets \( A \) and \( B \) such that \( A \cup B = X \).

Proof. (1 \( \rightarrow \) 2) Suppose that \((X, \eta)\) has a \(qD\)-division \( A \) and \( B \), then \( A \cup B = X \) and \( cl_\eta(A) \notin B \). Implies, \( cl_\eta(A) \subseteq B^c = X \setminus B \subseteq A \) but \( A \subseteq cl_\eta(A) \), so that \( A = cl_\eta(A) \). Therefore, \( A \) is a closed D-set. Similarly, we can see that \( B \) is also a closed D-set and \( A \nq B \) [by theorem 3.4]. Hence, the result.

(2 \( \rightarrow \) 3) Suppose that \((X, \eta)\) has a \(qD\)-division \( A \) and \( B \) such that \( A \) and \( B \) are closed D-sets, then \( A^c \) and \( B^c \) are open D-sets, \( A = B^c \) and \( B = A^c \). Therefore, \( A^c \cup B^c = X \) and \( A^c \nq B^c \). Hence, the result.

(3 \( \rightarrow \) 1) Since \( X = A \cup B \) such that \( A \nq B \) and \( A, B \) are open D-sets, then \( A \subseteq X \setminus B \) This implies that, \( cl_\eta(A) \subseteq X \setminus B \). Therefore, \( cl_\eta(A) \nq B \). Similarly, we have \( A \nq cl_\eta(B) \). Hence, \((X, \eta)\) has a \(qD\)-division.

**Theorem 3.21.** Let \((X, \eta)\) be a DTS. Then, the following are equivalent:

1. \((X, \eta)\) is D-connected,
2. \(X\) cannot be written as the union of two disjoint non-empty closed D-subsets,
3. \(X\) cannot be written as the union of two disjoint non-empty open D-subsets.

Proof. It follows from Theorem 3.19.

**Theorem 3.22.** Let \((X, \eta)\) be a DTS. Then, the following are equivalent:

1. \((X, \eta)\) is \(qD\)-connected,
2. \(X\) cannot be written as the union of two non quasi-coincident closed D-subsets,
3. \(X\) cannot be written as the union of two non quasi-coincident open D-subsets.

Proof. It follows from Theorem 3.20.

**Theorem 3.23.** Let \((X, \eta)\) be a DTS and let \( Y \) be a non-empty subset of \( X \). Then, if \( A \) and \( B \) are D-sets in \( Y \), then \( A \) and \( B \) are separated D-sets in \( Y \) if and only if \( A \) and \( B \) are separated D-sets in \( X \).

Proof. \( cl_\eta(A) \cap B = Y \cap cl_\eta(A) \cap B, B \subseteq Y \)

\[
= Y \cap B \cap cl_\eta(A) \\
= B \cap Y \cap cl_\eta(A) \\
= B \cap cl_\eta(A) \\
= \emptyset.
\]

Similarly, we have: \( cl_\eta(B) \cap A = \emptyset \).
Conversely, \( cl_{\eta_Y}(A) \cap B = Y \cap cl_{\eta_Y}(A) \cap B = Y \cap (cl_{\eta_Y}(A) \cap B) = Y \cap \emptyset = \emptyset \).

Similarly, we have: \( cl_{\eta_Y}(B) \cap A = \emptyset \). Hence, the result.

**Theorem 3.24.** Let \((X, \eta_Y)\) be a DTS and let \(Y\) be a non-empty subset of \(X\). Then, if \(A\) and \(B\) are D-sets in \(Y\), then \(A\) and \(B\) are q-separated D-sets in \(Y\) if \(A\) and \(B\) are q-separated D-sets in \(X\).

**Proof.** Suppose that \(A\) and \(B\) are qSD-sets in \(Y\), then \(cl_{\eta_Y}(A) \notin B\) and \(cl_{\eta_Y}(B) \notin A\). So that, \(cl_{\eta_Y}(A) \subseteq B^c\). Thus, \(Y \cap cl_{\eta_Y}(A) \subseteq Y \cap B^c\). It follows that, \(cl_{\eta_Y}(A) \subseteq B^c\).

Therefore, \(cl_{\eta_Y}(A) \notin B\).

Similarly, we have: \(cl_{\eta_Y}(B) \notin A\). Hence, the result.

**Lemma 3.25.** Let \((Y, \eta_Y)\) be a DT−subspace of a DTS \((X, \eta_Y)\). Then, if \((Y, \eta_Y)\) is a D-connected, then for every pair \(A\) and \(B\) of a separated D-subsets of \(X\) such that \(Y = A \cup B\), we have either \(A = \emptyset\) or \(B = \emptyset\).

**Proof.** Let \(A \neq \emptyset \neq B\) and \(Y = A \cup B\). Since \(A, B \subseteq Y\) and separated D-sets in \(X\), then they are separated D-sets in \(Y\) \([\text{by Theorem 3.23}]\). This implies that, \((Y, \eta_Y)\) is D-disconnected, which a contradiction. Hence, the result.

**Lemma 3.26.** Let \((Y, \eta_Y)\) be a DT−subspace of a DTS \((X, \eta_Y)\). Then, if \((Y, \eta_Y)\) is a qD−connected, then for every pair \(A\) and \(B\) of a q-separated D-subsets of \(X\) such that \(Y = A \cup B\), we have either \(A = \emptyset\) or \(B = \emptyset\).

**Proof.** Let \(A \neq \emptyset \neq B\) and \(Y = A \cup B\). Since \(A, B \subseteq Y\) and q-separated D-sets in \(X\), then they are q-separated D-sets in \(Y\) \([\text{by Theorem 3.24}]\). This implies that, \((Y, \eta_Y)\) is qD−disconnected, which a contradiction. Hence, the result.

**Theorem 3.27.** Let \((X, \eta)\) be a DTS and let \(Y\) be a non-empty subset of \(X\) such that \((Y, \eta_Y)\) is D-connected. Then, if \(A\) and \(B\) are separated D-subsets of \(X\) such that \(Y \subseteq A \cup B\), then \(Y \subseteq A\) or \(Y \subseteq B\).

**Proof.** Since \(Y \subseteq A \cup B\), then \(Y = Y \cap (A \cup B) = (Y \cap A) \cup (Y \cap B)\). By Theorem 3.23, \(Y \cap A\) and \(Y \cap B\) are separated D-sets of \(Y\). Since \((Y, \eta_Y)\) is D-connected, then \(Y \cap A = \emptyset\) or \(Y \cap B = \emptyset\) \([\text{by Lemma 3.25}]\). Therefore, \(Y \subseteq A\) or \(Y \subseteq B\).

**Theorem 3.28.** Let \((X, \eta)\) be a DTS and let \(Y\) be a non-empty subset of \(X\) such that \((Y, \eta_Y)\) is qD−connected. Then, if \(A\) and \(B\) are q-separated D-subsets of \(X\) such that \(Y \subseteq A \cup B\), then \(Y \subseteq A\) or \(Y \subseteq B\).

**Proof.** Since \(Y \subseteq A \cup B\), then \(Y = Y \cap (A \cup B) = (Y \cap A) \cup (Y \cap B)\). By Theorem 3.24 and Theorem 2.4, \(Y \cap A\) and \(Y \cap B\) are q-separated D-sets of \(Y\). Since \((Y, \eta_Y)\) is qD−connected, then \(Y \cap A = \emptyset\) or \(Y \cap B = \emptyset\) \([\text{by Lemma 3.26}]\). Therefore, \(Y \subseteq A\) or \(Y \subseteq B\).

**Theorem 3.29.** Let \(\{(X_\alpha, \eta_{\alpha}) : \alpha \in J\}\) be a family of non-empty D-connected subspaces of DTS \((X, \eta)\). Then, if \(\bigcap_{\alpha \in J} X_\alpha \neq \emptyset\), then \(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha}\) is D-connected subspace of \((X, \eta)\).
Proof. Let $Y = \bigcup_{\alpha \in J} X_\alpha$. Choose a D-point $x_\alpha \in Y$. Let $A$ and $B$ be D-division of $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$. Then, $x_\alpha \in A$ or $x_\alpha \in B$ without loss of generality, we may assume that $x_\alpha \in A$. For each $\alpha \in J$, since $(X_\alpha, \eta_{X_\alpha})$ is D-connected. It follows from Theorem 3.27 that, $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$. Therefore, we have $Y \subseteq A$, since $x_\alpha \in A$, and then $B = \emptyset$, which a contradiction. Hence, $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is a D-connected subspace of the DTS $(X, \eta)$.

**Theorem 3.30.** Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD$–connected subspaces of DTS $(X, \eta)$. Then, if $\bigcap_{\alpha \in J} X_\alpha \neq \emptyset$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is $qD$-connected subspace of a $(X, \eta)$.

**Proof.** Straightforward.

**Theorem 3.31.** Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty D-connected subspaces of DTS $(X, \eta)$. Then, if $X_\alpha \cap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of a $(X, \eta)$.

**Proof.** Fix an $\alpha_0 \in J$. For arbitrary $\beta \in J$, put $A_\beta = X_{\alpha_0} \cup X_\beta$. By Theorem 3.29, each $(A_\beta, \eta_{A_\beta})$ is D-connected. Then, $\{(A_\beta, \eta_{A_\beta}) : \beta \in J\}$ is a family non-empty D-connected subspaces of DTS $(X, \eta)$ and $\bigcap_{\beta \in J} A_\beta = X_{\alpha_0} \neq \emptyset$. Obvious, we have $\bigcup_{\alpha \in J} X_\alpha = \bigcup_{\beta \in J} A_\beta$. It follows from Theorem 3.29 that, $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is D-connected subspace of the DTS $(X, \eta)$.

**Theorem 3.32.** Let $\{(X_\alpha, \eta_{X_\alpha}) : \alpha \in J\}$ be a family of non-empty $qD$–connected subspaces of DTS $(X, \eta)$. Then, if $X_\alpha \cap X_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in J$, then $(\bigcup_{\alpha \in J} X_\alpha, \eta_{\bigcup_{\alpha \in J} X_\alpha})$ is $qD$–connected subspace of a $(X, \eta)$.

**Proof.** Straightforward.

**Theorem 3.33.** Let $(X, \eta)$ be a DTS and let $Y$ be a non-empty subset of $X$ such that $(Y, \eta_Y)$ is D-connected. Then, if $Y \subseteq A \subseteq cl_\eta(Y)$, then $(A, \eta_A)$ is a D-connected subspace of $(X, \eta)$. In particular, $(cl_\eta(Y), \eta_{cl_\eta(Y)})$ is a D-connected subspace of $(X, \eta)$.

**Proof.** Suppose that $(A, \eta_A)$ is a D-separated subspace of $(X, \eta)$, then $A$ has a D-separation $E$ and $G$ implies, $Y \subseteq E$ or $Y \subseteq G$ [by Theorem 3.27]. Without loss of generality, we may assume that $Y \subseteq E$, so $cl_\eta(Y) \subseteq cl_\eta(E)$. This completes the proof.

**Theorem 3.34.** Let $(X, \eta)$ be a DTS and let $Y$ be a non-empty subset of $X$ such that $(Y, \eta_Y)$ is $qD$–connected. Then, if $Y \subseteq A \subseteq cl_\eta(Y)$, then $(A, \eta_A)$ is a $qD$–connected subspace of $(X, \eta)$. In particular, $(cl_\eta(Y), \eta_{cl_\eta(Y)})$ is a $qD$–connected subspace of $(X, \eta)$.

**Proof.** Suppose that $(A, \eta_A)$ is a $qD$–connected subspace of $(X, \eta)$, then $A$ has a $qD$–separation $E$ and $\tilde{G}$ implies, $Y \subseteq E$ or $Y \subseteq \tilde{G}$ [by Theorem 3.28]. Without loss of generality, we may assume that $Y \subseteq E$, so $cl_\eta(Y) \subseteq cl_\eta(E)$. Thus $cl_\eta(Y) \notin \tilde{G}$. Otherwise, $\tilde{G} \subseteq A \subseteq cl_\eta(Y)$, Therefore, $(cl_\eta(Y))^c \notin \tilde{G}$, which a contradiction with $cl_\eta(Y) \subseteq cl_\eta(Y)$. This completes the proof.
**Theorem 3.35.** The image of D-connected under a D-continuous map are D-connected.

*Proof.* Let \((X, \eta)\) and \((Y, \tau)\) be two DTS, where \((X, \eta)\) is a D-connected and let \(f\) be a D-continuous from \(X\) onto \(Y\).

Suppose that \((Y, \tau)\) is a D-disconnected space, then \(\exists A, B \in \tau\) such that \(A \cap B = \emptyset\) and \(A \cup B = Y\). Implies \(A \subseteq B^c\), thus \(f^{-1}(A) \subseteq f^{-1}(B^c) = (f^{-1}(B))^c\). So that, \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\) and \(f^{-1}(A \cup B) = f^{-1}(Y)\). It follows that \(f^{-1}(A) \cup f^{-1}(B) = X\), i.e., \((X, \eta)\) is a D-connected, which a contradiction. Hence, \((Y, \tau)\) is a D-connected.

**Theorem 3.36.** The image of \(qD\)–connected under a D-continuous map are \(qD\)-connected.

*Proof.* Let \((X, \eta)\) and \((Y, \tau)\) be two DTS, where \((X, \eta)\) is a \(qD\)-connected and let \(f\) be a D-continuous from \(X\) onto \(Y\).

Suppose that \((Y, \tau)\) is a \(qD\)-disconnected space, then \(\exists A, B \in \tau\) such that \(A \not\subseteq B\) and \(A \cup B = Y\). Implies \(A \subseteq B^c\), thus \(f^{-1}(A) \subseteq f^{-1}(B^c) = (f^{-1}(B))^c\). So that, \(f^{-1}(A) \not\subseteq f^{-1}(B)\) and \(f^{-1}(A \cup B) = f^{-1}(Y)\). It follows that \(f^{-1}(A) \cup f^{-1}(B) = X\), i.e., \((X, \eta)\) is a \(qD\)-disconnected, which a contradiction. Hence, \((Y, \tau)\) is a \(qD\)-connected.

**Theorem 3.37.** Let \((X, \eta)\) be a DTS. Then, if \((X, \eta)\) is a D-disconnected space, then \((X, \pi_i), (i = 1, 2)\) are disconnected spaces.

*Proof.* Let \((X, \eta)\) be a D-disconnected. Then, \(\exists A, B \in \eta\) such that \((A_1, A_2) = A \cap B = (B_1, B_2) = \emptyset\) and \(A \cup B = X\). Thus, \(A_1 \cap B_1 = \emptyset, A_2 \cap B_2 = \emptyset, A_1 \cup B_1 = X\) and \(A_2 \cup B_2 = X\). Therefore, \(A_1 \cap B_i = \emptyset, A_i \cup B_i = X\) and \(A_i, B_i \in \pi_i, (i = 1, 2)\). Hence, \((X, \pi_1)\) and \((X, \pi_2)\) are disconnected spaces.

The following Example shows that the converse of Theorem 3.37 is not true in general.

**Example 3.38.** Let \(X = \{a, b, c\}\) and \(\pi_1 = \{\emptyset, X, \{a\}, \{b, c\}\}, \pi_2 = \{\emptyset, X, \{b\}, \{a, c\}\}\). Then, \((X, \pi_1)\) and \((X, \pi_2)\) are topological spaces and disconnected spaces. Since \(\eta = (\pi_1, \pi_2) = (\emptyset, X, \emptyset, X, \emptyset, \{b, c\}, \emptyset, \emptyset, \{a, c\})\), then \((X, \eta)\) is not D-connected.

**Theorem 3.39.** Let \((X, \eta)\) be a DTS. Then, if \((X, \eta)\) is a \(qD\)-disconnected space, then \((X, \pi_1)\) is disconnected space.

*Proof.* Let \((X, \eta)\) be a \(qD\)-disconnected space. Then, \(\exists A, B \in \eta\) such that \((A_1, A_2) = A \not\subseteq B = (B_1, B_2)\) and \(A \cup B = X\). Thus, \(A_1 \cap B_2 = \emptyset, A_1 \cup B_1 = X, B_1 \subseteq B_2\). So that \(A_1 \cap B_1 = \emptyset\) and \(A_1 \cup B_1 = X, A_1, B_1 \in \pi_1\). Hence, \((X, \pi_1)\) is a disconnected.

The following Example shows that the converse of Theorem 3.39 is not true in general.
Example 3.40. In Example 3.38, we see that: $(X, \pi_1)$ is topological space and disconnected. But, $(X, \eta)$ is not $qD$-disconnected.

Theorem 3.41. Let $(X, \eta), (X, \eta^*)$ be two $DTS$. Then, if $(X, \eta)$ is $D$-connected and $\eta^* \leq \eta$, then $(X, \eta^*)$ is also $D$-connected.

Proof. Suppose that $(X, \eta^*)$ is a $D$-disconnected and $\eta^* \leq \eta$, then $\exists A, B \in \eta^*$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Implies, $A, B \in \eta, A \cap B = \emptyset$ and $A \cup B = X$. Therefore, $(X, \eta)$ is a $D$-disconnected, which a contradiction. Hence, $(X, \eta^*)$ is $D$-connected.

Theorem 3.42. Let $(X, \eta), (X, \eta^*)$ be two $DTS$. Then, if $(X, \eta)$ is $qD$-connected and $\eta^* \leq \eta$, then $(X, \eta^*)$ is also $qD$-connected.

Proof. Suppose that $(X, \eta^*)$ is a $qD$-disconnected and $\eta^* \leq \eta$, then $\exists A, B \in \eta^*$ such that $A \not\subseteq B$ and $A \cup B = X$. Implies, $A, B \in \eta, A \not\subseteq B$ and $A \cup B = X$. Therefore, $(X, \eta)$ is a $qD$-disconnected, which a contradiction. Hence, $(X, \eta^*)$ is $qD$-connected.

Theorem 3.43. Every $qD$-connected space is $D$-connected.

Proof. Suppose that $(X, \eta)$ is a $D$-disconnected space, then $\exists A, B \in D(X)$ such that $cl_D(A) \cap B = \emptyset, A \cap cl_D(B) = \emptyset$ and $A \cup B = X$. It follows from Proposition 3.6 that $cl_D(A) \not\subseteq B, A \not\subseteq cl_D(B)$ and $A \cup B = X$. Therefore, $(X, \eta)$ is a $qD$-disconnected space, which a contradiction. Hence, $(X, \eta)$ is $D$-connected space.

Definition 3.44. The $DTS (X, \eta)$ is said to be:

1. $DC_1$-disconnected, if $(X, \eta)$ has a proper open and closed $D$-set in $X$.

2. $DC_1$-connected, if $(X, \eta)$ is not $DC_1$-disconnected.

Corollary 3.45. Let $(X, \eta)$ be a $DTS$ or stratified $DTS$. Then, if the only open and closed $D$-sets are $\emptyset, X$ in $(X, \eta)$ and $\emptyset, X$ and $(\emptyset, X)$ in stratified $DTS$, then $(X, \eta)$ is $qD$-connected.

Proposition 3.46. Every $DC_1$-connected space is $qD$-connected.

Proof. Suppose that $(X, \eta)$ is a $qD$-disconnected space, then $\exists A, B \not\subseteq \emptyset, B \not\subseteq \emptyset \in \eta$ such that $A \not\subseteq B$ and $A \cup B = X$. Implies $A \subseteq B^c$ and $B^c \subseteq A$, so that $A = B^c$. Therefore, $A$ is a proper open and closed $D$-set in $X$. Thus, $(X, \eta)$ be a $DC_1$-disconnected, which a contradiction. Hence, $(X, \eta)$ is $DC_1$-connected.

The converse of Proposition 3.46 is not true in general.

Example 3.47. Let $X = \{a, b, c\}$ and $\eta = (\emptyset, X, (\emptyset, \{b\}), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{c\}, \{c\}), (\{c\}, \{b, c\}), (\{a, c\}, \{a, c\}), (\{a, c\}, X), \eta^c = (\emptyset, X, (\{a, c\}, X), (\{b, c\}, \{b, c\}), (\{c\}, \{b, c\}), (\{a, b\}, \{a, b\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\emptyset, \{b\}))$. Then, $(X, \eta)$ is a $DTS$ and $qD$-connected space. But, $(X, \eta)$ is not $DC_1$-connected because, $\exists (\emptyset, \{b\}), (\{a\}, \{a, b\}), (\{c\}, \{b, c\})$ and $(\{a, c\}, X)$ are open and closed $D$-sets.
Example 3.48. Let $X = \{a, b, c\}$ and $\eta = \emptyset, X, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{c\}), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, b, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, b, c\})$.

$$\eta^c = \emptyset, X, (\{b, c\}, X), (\{a, c\}, X), (\{a, b\}, X), (\{a\}, X), (\{b\}, X), (\{c\}, X), (\emptyset, X), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a\}, X), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X), (\emptyset, \{a\}, X), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X), (\emptyset, \{a\}, X), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X), (\emptyset, \{a\}, X), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X).$$

Then, $(X, \eta)$ is stratified $DTS$ and $qD$-connected space. But, $(X, \eta)$ is not $DC_1$-connected because, $\exists (\emptyset, \{c\}), (\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a\}, X), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X)$, and $\{a, b\}, X$ are open and closed D-sets.

Example 3.49. From Example 3.48 $(X, \eta)$ is stratified $DTS$ and $qD$-connected, but $\exists (\emptyset, \{c\}), (\{a\}, X), (\{a\}, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{b, c\}), (\emptyset, \{c\}, X), (\{a\}, \{a, c\}), (\emptyset, \{b\}, X), (\emptyset, \{c\}, X)$, and $\{a, b\}, X$ are open and closed D-sets.

Corollary 3.50. For a $DTS$ $(X, \eta)$ we have the following implication:

$$DC_1\text{-connected} \rightarrow qD\text{-connected} \rightarrow D\text{-connected}.$$ 

Definition 3.51. The $DTS$ $(X, \eta)$ is said to be:

1. Strongly double connected (strongly $SD$-connected, for short), if there exist no non-empty closed D-sets $A, B \in X$ such that $A \cap B = \emptyset$.

2. Strongly $SD$-disconnected, if $(X, \eta)$ is not strongly $SD$-connected.

Proposition 3.52. $(X, \eta)$ is strongly $SD$-connected if and only if there exist no open D-sets $A, B$ in $X$ such that $A \neq X \neq B$ and $A \cup B = X$.

Proof. Let $A, B$ be open D-sets in $X$ such that $A \neq X \neq B$. If we take $C = A^c$ and $D = B^c$, then $C$ and $D$ become closed D-sets in $X$ and $C \neq \emptyset \neq D$. Consequently, $C \cap D = \emptyset$, which is a contradiction.

Conversely, it is obvious.

Proposition 3.53. Strongly $SD$-connectedness does not imply $DC_1$-connectedness, and $DC_1$-connectedness does not imply $STD$-connectedness.

Example 3.54. In Example 3.7, we see that: $(X, \eta)$ is strongly $SD$-connected, but it is not $DC_1$-connected, for $\exists (\emptyset, \{b\})$ is both open and closed D-set.

Example 3.55. In Example 3.3, we see that: $(X, \eta)$ is $DC_1$-connected, but it is not strongly $SD$-connected, for

$$\exists (\{a\}, \{a, b\}), (\{b, c, d\}, X) \in \eta \text{ and } (\{a\}, \{a, b\}) \cup (\{b, c, d\}, X) = X.$$ 

Definition 3.56. Let $(X, \eta)$ be a $DTS$ and $Y \subseteq X$ with $x \in DP(Y)$. The union of all D-connected subsets of $Y$ containing the D-point $x_s$ is called double component (D-component, for short) of $Y$ with respect to $x_s$, denoted by $C(Y, x_s)$, i.e., $C(Y, x_s) = \cup\{A \subseteq Y : x_s \in A \text{ and } A \text{ is a } D\text{-connected set}\}$. 
Remark 3.57. The D-component \( C(Y, x_t) \) is the largest D-connected subset of \( Y \) containing \( x_t \).

Definition 3.58. Let \((X, \eta)\) be a \( DTS \) and \( Y \subseteq X \) with \( x_t \in DP(Y) \), \( x_t \in q cl(x_{\bar{t}}) \). Then, the union of all \( qD \)–connected subsets of \( Y \) containing the D-point \( x_t \) is called quasi-coincident double component (\( qD \)–component, for short) of \( Y \) with respect to \( x_t \), denoted by \( Cq(Y, x_t) \), i.e., \( Cq(Y, x_t) = \cup \{A \subseteq Y : x_t \in A \text{ and } A \text{ is a } qD \text{–connected set}\} \).

Remark 3.59. The \( qD \)–component \( Cq(Y, x_t) \) is the largest \( qD \)–connected subset of \( Y \) containing \( x_t \).

Theorem 3.60. Every D-component of a \( DTS \) is a closed D-set.

Proof. Let \((X, \eta)\) be a \( DTS \) and let \( C(Y, x_t) \) be a D-component of the \( DTS \) \((X, \eta)\) with respect to an arbitrary D-point \( x_t \in DP(X) \). Then, \( C(Y, x_t) \) is a D-connected subset of \( X \) [Theorem 3.29]. Also, by Theorem 3.33 \( cl_\eta (C(Y, x_t)) \) is D-connected subset of \( X \) containing \( x_t \), then \( cl_\eta (C(Y, x_t)) \subseteq C(Y, x_t) \). But, \( C(Y, x_t) \subseteq cl_\eta (C(Y, x_t)) \). Hence, \( C(Y, x_t) = cl_\eta (C(Y, x_t)) \), which shows that the D-component \( C(Y, x_t) \) is a closed D-set.

Theorem 3.61. Every \( qD \)–component of a \( DTS \) is a closed D-set.

Proof. Let \((X, \eta)\) be a \( DTS \) and let \( Cq(Y, x_t) \) be a \( qD \)–component of the \( DTS \) \((X, \eta)\) with respect to an arbitrary D-point \( x_t \in DP(X) \). Then, \( Cq(Y, x_t) \) is a \( qD \)–connected subset of \( X \) [Theorem 3.30]. Also, by Theorem 3.34 \( cl_\eta (Cq(Y, x_t)) \) is \( qD \)–connected subset of \( X \) containing \( x_t \), then \( cl_\eta (Cq(Y, x_t)) \subseteq Cq(Y, x_t) \). But, \( Cq(Y, x_t) \subseteq cl_\eta (Cq(Y, x_t)) \). Hence, \( Cq(Y, x_t) = cl_\eta (Cq(Y, x_t)) \), which shows that the \( qD \)–component \( Cq(Y, x_t) \) is a closed D-set.

Theorem 3.62. Let \((X, \eta)\) be a \( DTS \). Then, each D-point in \( X \) is contained in exactly one D-component of \( X \).

Proof. Let \( x_t \in X \) and consider the collection:
\[
C = \{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } D \text{–connected set}\}.
\]
Then, we have:

1. \( C \neq \emptyset \), for the D-point \( x_t \) is a D-connected subset of \( X \). Then, \( x_t \in C \).
2. \( \bigcap \{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } D \text{–connected set}\} = \emptyset \). Since \( x_t \in Y \forall Y \in C \).
3. \( \bigcup \{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } D \text{–connected set}\} \), having non null double intersection, is D-connected subset of \( X \) containing \( x_t \).
4. \( \bigcup \{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } D \text{–connected set}\} \) is the largest D-connected subset of \( X \) containing \( x_t \), which is the D-component \( C(X, x_t) \) of \( X \) with respect to \( x_t \) and containing \( x_t \) from Definition 3.56.
Now, suppose that $C^*(X, x_t)$ be another D-component containing $x_t$, then $C^*(X, x_t)$ is D-connected subset of $X$ containing $x_t$. Since $C(X, x_t)$ is D-component containing $x_t$, then $C^*(X, x_t) \subseteq C(X, x_t)$. Again, since $C^*(X, x_t)$ is D-component containing $x_t$, then $C(X, x_t) \subseteq C^*(X, x_t)$. Therefore, $C(X, x_t) = C^*(X, x_t)$. Hence, $x_t$ is contained in exactly one D-component of $X$.

**Theorem 3.63.** Let $(X, \eta)$ be a DTS. Then, each D-point in $X$, $x_\perp \ q cl_\eta(x_\perp)$, is contained in exactly one $qD$–component of $X$.

**Proof.** Let $x_t \in X$ and consider the collection: $Cq = \{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } qD \text{– connected set}\}$.

Then, we have:

1. $Cq \neq \emptyset$, for the D-point $x_t$ is a $qD$–connected subset of $X$, $x_\perp \ q cl_\eta(x_\perp)$. Then, $x_t \in Cq$.

2. $\bigcap\{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } qD \text{– connected set}\} \neq \emptyset$. Since $x_t \in Y, \forall Y \in C$.

3. $\bigcup\{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } qD \text{– connected set}\}$, having non null double intersection, is $qD$–connected subset of $X$ containing $x_t$.

4. $\bigcup\{Y \subseteq X : x_t \in Y \text{ and } Y \text{ is a } qD \text{– connected set}\}$ is the largest $qD$–connected subset of $X$ containing $x_t$, which is the $qD$–component $Cq(x_t)$ of $X$ with respect to $x_t$, and containing $x_t$, from Definition 3.58.

Now, suppose that $Cq^*(X, x_t)$ be another $qD$–component containing $x_t$, thus $Cq^*(X, x_t)$ is $qD$–connected subset of $X$ containing $x_t$. Since $Cq(X, x_t)$ is $qD$–component containing $x_t$, then $Cq^*(X, x_t) \subseteq Cq(X, x_t)$. Again, since $Cq^*(X, x_t)$ is $qD$–component containing $x_t$, then $Cq(X, x_t) \subseteq Cq^*(X, x_t)$. Therefore, $Cq(X, x_t) = Cq^*(X, x_t)$. Hence, $x_t$ is contained in exactly one $qD$–component of $X$.

**Theorem 3.64.** Let $(X, \eta)$ be a DTS. Then, any two D-components with respect to two disjoint D-points in $X$ are either disjoint or identical.

**Proof.** Let $C(X, x_t)$ and $C(X, y_r)$ be two D-components of the DTS $(X, \eta)$ with respect to the D-points $x_t, y_r$ in $X$, and $x_t \cap y_r = \emptyset$. Then, if $C(X, x_t) \cap C(X, y_r) = \emptyset$, then we are done. So let $C(X, x_t) \cap C(X, y_r) \neq \emptyset$. We may choose $z \in C(X, x_t) \cap C(X, y_r)$. Clearly, $z \in C(X, x_t)$ and $z \in C(X, y_r)$. Thus, $C(X, x_t) = C(X, y_r)$ [from Theorem 3.62]. Therefore, $C(X, x_t)$ and $C(X, y_r)$ are identical. This completes the proof.

**Theorem 3.65.** Let $(X, \eta)$ be a DTS. Then, any two $qD$–components with respect to two disjoint D-points in $X$ are either not quasi-coincident or identical.

**Proof.** Let $Cq(X, x_t)$ and $Cq(X, y_r)$ be two $qD$–components of the DTS $(X, \eta)$ with respect to the D-points $x_t, y_r$ in $X$, $x_\perp \ q cl_\eta(x_\perp)$, $y_\perp \ q cl_\eta(y_\perp)$ and $x_\perp \ y_\perp = \emptyset$. Then, if $Cq(X, x_t) \neq Cq(X, y_r)$, then we are done. So let $Cq(X, x_t) \cap Cq(X, y_r) \neq \emptyset$. We may choose $z \in Cq(X, x_t) \cap Cq(X, y_r)$. Clearly, $z \in Cq(X, x_t)$ and $z \in Cq(X, y_r)$, thus $Cq(X, x_t) = Cq(X, y_r)$ [from Theorem 3.63]. Therefore, $Cq(X, x_t)$ and $Cq(X, y_r)$ are identical. This completes the proof.
**Definition 3.66.** A DTS $(X, \eta)$ is said to be a quasi-coincident hyperconnected (q-hyperconnected, for short) if every pair of non null proper open D-sets $A$ and $B$ are quasi-coincident, i.e.,

$(X, \eta)$ is said to be q-hyperconnected if $\forall A, B \in \eta$, we have $A \not\sim B$.

**Theorem 3.67.** Every q-hyperconnected DTS is qD-connected.

**Proof.** Suppose that $(X, \eta)$ is qD-disconnected, then there exist two open D-sets $A$ and $B$ such that $A \not\sim B$ [Theorem 3.20]. Hence, $(X, \eta)$ is not q-hyperconnected DTS, which a contradiction. Thus, $(X, \eta)$ is qD-connected.

**Remark 3.68.** The converse of Theorem 3.67 is not true in general, as shown in the following Examples.

**Example 3.69.** In Example 3.7, we see that: The DTS $(X, \eta)$ is qD-connected, but it is not q-hyperconnected, for $(\{a\}, \{a\}) \not\sim (\emptyset, \{b\})$.

**Example 3.70.** Let $X = \{a, b\}$ and $\eta = \{\emptyset, X, (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, X)\}$. Then, $\eta$ defines a stratified double topology on $X$. Hence, the DTS $(X, \eta)$ is qD-connected, but it is not q-hyperconnected, for $(\emptyset, \{a\}) \not\sim (\emptyset, \{b\})$.

**Theorem 3.71.** Let $(X, \eta)$ be a DTS. Then, the following are equivalent:

1. $(X, \eta)$ is q-hyperconnected,
2. $A$ is D-dense, $\forall A \in \eta$,
3. $A$ is D-dense or D-nowhere dense, $\forall A \in D(X)$.

**Proof.** (1) $\rightarrow$ (2) Suppose that $\exists B \in \eta$ such that $B$ is not D-dense in $X$, thus $cl_\eta(B) \neq X$. Hence, $X \setminus cl_\eta(B)$ and $B$ are not quasi-coincident [by Proposition 3.3], which a contradiction with q-hyperconnected of $(X, \eta)$.

(2) $\rightarrow$ (3) Suppose that $B$ is not D-nowhere dense, then $int_\eta(cl_\eta(B)) \neq \emptyset$. So by (2), $cl_\eta(int_\eta(cl_\eta(B))) = X$. Since $cl_\eta(int_\eta(cl_\eta(B))) \subseteq cl_\eta(B)$, then $cl_\eta(B) = X$. Hence, $B$ is D-dense.

(3) $\rightarrow$ (1) Suppose that $A \not\sim B$, for some non-empty open D-subsets $A, B$ of $X$, then $cl_\eta(A) \not\sim B$ [by Theorem 2.7], and $A$ is not D-dense. Since $A \in \eta$, then $\emptyset \neq A \subseteq int_\eta(cl_\eta(A))$, which a contradiction with (3). Hence, the result.

**Remark 3.72.** If the DTS $(X, \eta)$ is a stratified, then the Theorems 3.19, 3.20, ..., 3.71 and Propositions 3.2, ..., 3.53 are satisfied.

**References**


