SOME CONTRIBUTION OF SOFT RUGULAR OPEN SETS TO SOFT SEPARATION AXIOMS IN SOFT QUAD TOPOLOGICAL SPACES

Arif Mehmood Khattak\(^1\)  \(<\text{mehdaniyal@gmail.com}>\)  
Asad Zaighum\(^1\)  \(<\text{asad.zaigum@riphah.edu.pk}>\)  
Muhammad Ishfaq\(^1\)  \(<\text{muhammad.ishfaqtk@gmail.com}>\)  
Shamoona Jbeen\(^2\)  \(<\text{shamoonafazal@yahoo.com}>\)  
Iqrar Hussain\(^3\)  \(<\text{iqrarktk13@gmail.com}>\)

\(^1\)Department of Mathematics and Statistics, Riphah International University, Pakistan  
\(^2\)School of Mathematics and System Sciences, Beihang University Beijing, China  
\(^3\)Department of Mathematics Govt., Post Graduate College Karak, Pakistan

Abstract-Our main interest in this study is to look for soft regular separations axioms in soft quad topological spaces. We talk over and focus our attention on soft regular separation axioms in soft quad topological spaces with respect to ordinary points and soft points. Moreover study the inherited characteristics at different angles with respect to ordinary points and soft points. Some of their central properties in soft quad topological spaces are also brought under examination.

Keywords- Soft sets, soft topology, soft regular open set, soft regular closed set, soft quad topological space, soft R\(_q\)T\(_a\) structure, R-soft qT\(_2\) structure, soft R-qT\(_2\) structure, soft R-qT\(_3\) structure and soft R-qT\(_4\) structure.

1 Introduction

In real life condition the complications in economics, engineering, social sciences, medical science etc. We cannot handsomely use the old-fashioned classical methods because of different types of uncertainties existing in these problems. To finish out these complications, some types of theories were put forwarded like theory of Fuzzy set, intuitionistic fuzzy set, rough set and bi polar fuzzy sets, inwhich we can safely use a mathematical methods for dealing with uncertainties. But, all these theories have their inherent worries. To overcome these difficulties in the year 1999, Russian scholar Molodtsov [4]introduced the idea of soft set as a new mathematical methods to deal with uncertainties. This is free from the above difficulties. Kelly [5] studied Bi topological spaces and discussed different results. Tapi et al. beautifully discussed separation axioms.
in quad topological spaces. Hameed and Abid discussed separation axioms in Tri-
topological spaces.

Recently, in 2011, Shabir and Naz [7] initiated the idea of soft topological space and
discussed different results with respect to ordinary points, they beautifully defined soft
topology as a collection of $\tau$ of soft sets over $X$. They also defined the basic concept of
soft topological spaces such as open set and closed soft sets, soft nbd of a point, soft
separation axioms, soft regular and soft normal spaces and published their several
performances. Soft separation axioms are also discussed at detail. Aktas and Cagman [9]
discussed soft sets and soft groups. Chen [10] discovered the parameterization reduction
of soft sets and its applications. Feng et al. [11] studied soft semi rings and its
applications. In the recent years, many interesting applications of soft sets theory and
soft topology have been discussed at great depth [12,13,14,15,16,17,18,19,20,21,22] Kandil
et al. [25] explained soft connectedness via soft ideal developed soft set theory. Kandil
et al. [27] launched soft regularity and normality based on semi open soft sets and
soft ideals.

In [28,29,30,31,32,33,34,35,36] discussion is launched soft semi Hausdorff spaces via
soft ideals, semi open and semi closed sets, separation axioms, decomposition of some type
supra soft sets and soft continuity are discussed. Hussain and Ahmad [51] defined soft
points, soft separation axioms in soft topological spaces with respect to soft points and
used it in different results. Kandil et al. [52] studied soft semi separation axioms and
some types of soft functions and their characteristics.

In this present paper, concept of soft regular separation axioms in soft quad topological
spaces is broadcasted with respect to ordinary and soft points.

Many mathematicians made discussion over soft separation axioms in soft topological
spaces at full length with respect to soft open set, soft $b$-open set, soft semi-open, soft
$\alpha$-open set and soft $\beta$-open set. They also worked over the hereditary properties of
different soft topological structures in soft topology. In this present article h and is tried
and work is encouraged over the gap that exists in soft quad-topology related to soft
regular $R-QT_0$, soft regular $R-QT_1$, soft regular $R-QT_2$, soft regular $R-QT_3$ and soft
regular-$QT_4$ structures. Some propositions in soft quid topological spaces are discussed
with respect to ordinary points and soft points. When we talk about distance between the
points in soft topology then the concept of soft separation axioms will auto medically
come in force. That is why these structures are catching our attentions. We hope that
these results will be valuable for the future study on soft quad topological spaces to
accomplish general framework for the practical applications and to solve the most
complicated problems containing doubts in economics, engineering, medical, environment
and in general mechanic systems of various varieties. In upcoming these beautiful soft
topological structures may be extended in to soft $n$-topological spaces provided $n$ is even.

2. Preliminaries

The following Definition s which are pre-requisites for present study
Definition 1 [4] Let $X$ be an initial universe of discourse and $E$ be a set of parameters. Let $P(X)$ denotes the power set of $X$ and $A$ be a non-empty sub-set of $E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $FA \rightarrow P(X)$

In other words, a set over $X$ is a parameterized family of sub set of universe of discourse $X$. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set $(F, A)$ and if $e \notin A$ then $F(e) = \phi$ that is $F = \{F(e) \in A \subseteq E, FA \rightarrow P(X)\}$ the family of all these soft sets over $X$ denoted by $SS(X)_A$

Definition 2 [4] Let $F_A, G_B \in SS(X)_E$ then $F_A \subseteq G_B$ is a soft subset of $G_B$ denoted by $F_A \subseteq G_B$, if
1. $A \subseteq B$ and
2. $F(e) \subseteq G(e), \forall e \in A$

In this case $F_A$ is said to be a soft subset of $G_B$ and $G_B$ is said to be a soft super set $F_A, G_B \supseteq F_A$

Definition 3 [6] Two soft subsets $F_A$ and $G_B$ over a common universe of discourse set $X$ are said to be equal if $F_A$ is a soft subset of $G_B$ and $G_B$ is a soft subset of $F_A$

Definition 4 [6] The complement of soft subset $(F, A)$ denoted by $(F, A)^c$ is defined by $(F, A)^c = (F^c, A) F^c \rightarrow P(X)$ is a mapping given by $F^c(e) = U - F(e), \forall e \in A$ and $F^c$ is called the soft complement function of $F$. Clearly $(F^c)^c$ is the same as $F$ and $(F, A)^c)^c = (F, A)$

Definition 5 [7] The difference between two soft subset $(G, E)$ and $(G, E)$ over common of universe discourse $X$ denoted by $(G, E) \setminus (G, E)$ is defined as $F(e) \setminus G(e)$ for all $e \in E$

Definition 6 [7] Let $(G, E)$ be a soft set over $X$ and $x \in X$ We say that $x \in (F, E)$ and read as $x$ belong to the soft set $(F, E)$ whenever $x \in F(e), \forall e \in E$. The soft set $(F, E)$ over $X$ such that $F(e) = \{x\}, \forall e \in E$ is called sing Let on soft point and denoted by $x, or (x, E)$

Definition 7 [6] A soft set $(F, A)$ over $X$ is said to be Null soft set denoted by $\emptyset$ or $\emptyset_A$ if $\forall e \in A, F(e) = \emptyset$

Definition 8 [6] A soft set $(F, A)$ over $X$ is said to be an absolute soft denoted by $A$ or $X_A$ if $\forall e \in A, F(e) = X$.

Clearly, we have $X_A^c = \emptyset_A$ and $\emptyset_A^c = X_A$

Definition 9 [38] The soft set $(F, A) \in SS(X)_A$ is called a soft point in $X_A$, denoted by $e_F$, if for the element $e \in A, F(e) \neq \{x\}$ and $F(e^i) = \phi$ if for all $e^i \in A - \{e\}$.

A soft point is an element of a soft set $F_A$. The class of all soft sets over $U$ is denoted by $SS(U)$.
For Example \( U = \{u_1, u_2, u_3\}, E = \{x_1, x_2, x_3\}, A = \{x_1, x_2\} \) and \( F_A = \{(x_1, \{u_1, u_2\})\} \), \( (x_2, \{u_2, u_3\})\). Then \( F_{A_1} = \{(x_1, \{u_1\})\}, F_{A_2} = \{(x_1, \{u_2\})\}, F_{A_3} = \{(x_1, \{u_2, u_2\})\} \), \( F_{A_4} = \{(x_2, \{u_2\})\}, F_{A_5} = \{(x_2, \{u_3\})\}, F_{A_6} = \{(x_2, \{u_1, u_3\})\}\), \( F_{A_7} = \{(x_1, \{u_2\}, (x_2, \{u_2\})\}, F_{A_8} = \{(x_1, \{u_1\}, (x_2, \{u_3\})\}, F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}, F_{A_{10}} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}, F_{A_{11}} = \{(x_1, \{u_2\}), (x_2, \{u_3\})\}, F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}, F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}, F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\})\}, F_{A_{15}} = F_A, F_{A_{16}} = F_0, \) are all soft sub sets of \( F_A \).

**Definition 10** [38] The soft point \( e_f \) is said to be in the soft set \((G, A)\), denoted by \( e_f \in (G, A) \) if for the element \( e \in A, F(e) \subseteq G(e) \).

**Definition 11** [6] The union of two soft sets \((F, A)\) and \((G, B)\) over the common universe of discourse \( X \) is the soft set \((H, C)\), where, \( C = A \cup B \) and for all \( e \in C \)

\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A - B \\
G(e) & \text{if } e \in (B - A) \\
F(e) \cup G(e), & \text{if } e \in A \cap B
\end{cases}
\]

Written as \((F, A) \cup (G, B) = (H, C)\)

**Definition 12** [6] The intersection \((H, C)\) of two soft sets \((F, A)\) and \((G, B)\) over common universe \( X \), denoted \((F, A) \cap (G, B)\) is defined as \( C = A \cap B \) and \( H(e) = F(e) \cap G(e), \forall e \in C \)

**Definition 13** [41] Two soft sets \((G, A), (H, A)\) in \( SS(X) \) are said to be soft disjoint, written \((G, A) \cap (H, A) = \emptyset_A\) if \( G(e) \cap H(e) = \emptyset \) for all \( e \in A \).

**Definition 14** [38] The soft point \( e_G, e_H \) in \( X_A \) are disjoint, written \( e_G \neq e_H \) if their corresponding soft sets \((G, A)\) and \((H, A)\) are disjoint.

**Definition 15** [2] Let \((F, E)\) be a soft set over \( X \) and \( Y \) be a non-empty sub set of \( X \). Then the sub soft set of \((F, E)\) over \( Y \) denoted by \((Y_F, E)\), is defined as follow \( Y_F(\alpha) = Y \cap F(\alpha), \forall \alpha \in E \) in other words \( (Y, F, E) = Y \cap (F, E) \).

**Definition 16** [3] Let \( \tau \) be the collection of soft sets over \( X \), then \( \tau \) is said to be a soft topology on \( X \), if

1. \( \emptyset, X \) belong to \( \tau \)
2. The union of any number of soft sets in \( \tau \) belongs to \( \tau \)
3. The intersection of any two soft sets in \( \tau \) belong to \( \tau \)

The trip Let \((X, F, E)\) is called a soft topological space.
**Definition 17** [1] Let \((X, F, E)\) be a soft topological space over \(X\), then the member of \(\mathcal{I}\) are said to be soft open sets in \(X\).

**Definition 18** [1] Let \((X, F, E)\) be a soft topological space over \(X\). A soft set \((F, A)\) over \(X\) is said to be a soft closed set in \(X\) if its relative complement \((F, E)^c\) belong to \(\mathcal{I}\).

**Definition 19** [42] A soft set \((A, E)\) in a soft topological space \((X, \tau, E)\) will be termed soft regular open set denoted as \(S(R, U(\tau, E))\) if and only if there exists a soft open set \((F, E) = \text{int}(\text{cl}(F, E))\) and soft regular closed set if set \((F, E) = \text{cl}(\text{in}(F, E))\) denoted by as \(S(R, C(\tau, E))\) in short h.

3. Soft Regular Separation Axioms of Soft Quad Topological Spaces

In this section we introduced soft regular Separation Axioms in soft Quad topological space with respect to ordinary points and discussed some results with respect to these points in detail.

**Definition 20** Let \(Y, F, E, Z, \mathcal{I}, Y, \mathcal{I}, Z, \mathcal{I}, Y, \mathcal{I}, Z, \mathcal{I}, \) be four different soft topologies on \(X\). Then \(\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4, X, \mathcal{I}, Z, \mathcal{I}, \) is called a soft quad topological space. The soft four topologies \(X, \mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4, X, \mathcal{I}, \) and \(X, \mathcal{I}^4, \mathcal{I}, \) are independently satisfying the axioms of soft topology. The members of \(\mathcal{I}^1\) are called \(\mathcal{I}^1\) soft open set and complement of \(\mathcal{I}^1\) soft open set is called \(\mathcal{I}^2\) soft closed set. Similarly, the members of \(\mathcal{I}^2\) are called \(\mathcal{I}^2\) soft open set and the complement of \(\mathcal{I}^2\) soft open sets are called \(\mathcal{I}^3\) soft closed set. The members of \(\mathcal{I}^3\) are called \(\mathcal{I}^3\) soft open set and complement of \(\mathcal{I}^3\) soft open set is called \(\mathcal{I}^4\) soft closed set and the members of \(\mathcal{I}^4\) are called \(\mathcal{I}^4\) soft open set.

**Definition 21** Let \(X, Y, C, \mathcal{I}, T, \mathcal{I}, \) be a soft quad topological space over \(X\) and \(Y\) be a non-empty subset of \(X\). Then \(\mathcal{I}^1_Y = \{(Y, F, E) (F, E) \in \mathcal{I}^1\}, \mathcal{I}^2_Y = \{(Y, G, E) (G, E) \in \mathcal{I}^2\}, \mathcal{I}^3_Y = \{(Y, H, E) (H, E) \in \mathcal{I}^3\}\) and \(\mathcal{I}^4_Y = \{(I, E) (I, E) \in \mathcal{I}^4\}\) are said to be the relative topological on \(Y\). Then \(\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4, X, \mathcal{I}, \mathcal{I}, \) is called relative soft quad-topological space \(X, \mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4, X, \mathcal{I}, \mathcal{I}, \).
Let \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) be a soft quad topological space over \( X \), where \( (\bar{X}, \bar{\tau}, \bar{E}) \), \( (\bar{X}, \bar{\tau}, \bar{E}) \), \( (\bar{X}, \bar{\tau}, \bar{E}) \) and \( (\bar{X}, \bar{\tau}, \bar{E}) \) be four different soft topologies on \( \bar{X} \).

Then a sub set \( (F, \bar{E}) \) is said to be quad-open (in short h and q-open) if \( (F, \bar{E}) \subseteq \bar{\tau} \cup \bar{\tau} \cup \bar{\tau} \cup \bar{\tau} \) and its complement is said to be soft q-closed.

3.1 Soft Regular Separation Axioms of Soft Quad Topological Spaces with Respect to Ordinary Points

In this section we introduced soft semi separation axioms in soft quad topological space with respect to ordinary points and discussed some attractive results with respect to these points in detail.

**Definition 22** Let \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) be a soft quad topological space over \( \bar{X} \) and \( x, y \in \bar{X} \) such that \( x \neq y \). If we can find soft q-open sets \( (F, \bar{E}) \) and \( (G, \bar{E}) \) such that \( x \in (F, \bar{E}) \) and \( y \notin (F, \bar{E}) \) or \( y \in (G, \bar{E}) \) and \( x \notin (G, \bar{E}) \) then \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) is called soft \( qT_0 \) space.

**Definition 23** Let \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) be a soft quad topological space over \( X \) and \( x, y \in X \) such that \( x \neq y \). If we can find two soft q-open sets \( (F, \bar{E}) \) and \( (G, \bar{E}) \) such that \( x \in (F, \bar{E}) \) and \( y \notin (F, \bar{E}) \) and \( y \in (G, \bar{E}) \) and \( x \notin (G, \bar{E}) \) then \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) is called soft \( qT_1 \) space.

**Definition 24** Let \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) be a soft quad topological space over \( X \) and \( x, y \in X \) such that \( x \neq y \). If we can find two q-open soft sets such that \( x \in (F, \bar{E}) \) and \( y \in (G, \bar{E}) \) moreover \( (F, \bar{E}) \cap (G, \bar{E}) = \phi \). Then \( (X, \tau_1, \tau_2, \tau_3, \tau_4, E) \) is called a soft \( qT_2 \) space.

**Definition 25** Let \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) be a soft topological space \( (G, \bar{E}) \) be q-closed soft set in \( X \) and \( x \in X \) such that \( x \notin (G, \bar{E}) \). If there occurs soft q-open sets \( (F_1, \bar{E}) \) and \( (F_2, \bar{E}) \) such that \( x \in (F_1, \bar{E}) \), \( (G, \bar{E}) \subseteq (F_2, \bar{E}) \) and \( (F_1, \bar{E}) \cap (F_2, \bar{E}) = \phi \). Then \( (\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}) \) is called soft q-regular spaces. A soft q-regular \( qT_1 \) space is called soft \( qT_3 \) space.
Then \( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \) is called a soft q-regular spaces. A soft q-regular \( T_1 \) Space is called soft \( qT_3 \) space.

**Definition 26** \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) be a soft quad topological space \((F_1, E), (G, \bar{E})\) be closed soft sets in \( X \) such that \( (F, E) \cap (G, \bar{E}) = \emptyset \) if there exists q-open soft sets \((F_1, \bar{E})\) and \((F_2, \bar{E})\) such that \( (F, E) \subseteq (F_1, \bar{E}), (G, \bar{E}) \subseteq (F_2, \bar{E}) \) and \((F_1, \bar{E}) \cap (F_2, \bar{E}) = \emptyset \). Then \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) is called a q-normal space. A soft q-normal \( qT_3 \) Space is called soft \( qT_4 \) Space.

**Definition 27** Let \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) be a soft topological space over \( X \) and \( e_G, e_H \in X_A \) such that \( e_G \neq e_H \) if there can happen at least one soft q-open set \((F_1, \bar{A})\) or \((F_2, \bar{A})\) such that \( e_G \in (F_1, \bar{A}), e_H \notin (F_1, \bar{A}) \) or \( e_H \in (F_2, \bar{A}), e_G \notin ((F_2, \bar{A}) \) then \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) is called a soft \( qT_0 \) space.

**Definition 28** Let \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) be a soft topological spaces over \( X \) and \( e_G, e_H \in X_A \) such that \( e_G \neq e_H \) if there can happen soft q-open sets \((F_1, \bar{A})\) and \((F_2, \bar{A})\) such that \( e_G \in (F_1, \bar{A}), e_H \notin (F_1, \bar{A}) \) and \( e_H \in (F_2, \bar{A}), e_G \notin ((F_2, \bar{A}) \) then \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) is called soft \( qT_1 \) space.

**Definition 29** Let \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) be a soft topological space over \( X \) and \( e_G, e_H \in X_A \) such that \( e_G \neq e_H \) if there can happen soft q-open sets \((F_1, \bar{A})\) and \((F_2, \bar{A})\) such that \( e_G \in (F_1, \bar{A}), e_H \notin (F_1, \bar{A}) \) and \( e_H \in (F_2, \bar{A}) \cap (F_2, \bar{A}) = \emptyset \) . Then \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) is called soft \( qT_2 \) space.

**Definition 30** Let \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) be a soft topological space \((G, \bar{E})\) be q-closed soft set in \( X \) and \( e_G \in X_A \) such that \( e_G \notin (G, \bar{E}) \) . if there occurs soft q-open sets \((F_1, \bar{E})\) and \((F_2, \bar{E})\) such that \( e_G \in (F_1, \bar{E}), (G, \bar{E}) \subseteq (F_2, \bar{E}) \) and \((F_1, \bar{E}) \cap (F_2, \bar{E}) = \emptyset \). Then \( \left( \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right) \) is called soft q-regular spaces. A soft q-regular \( qT_3 \) Space is called soft \( qT_3 \) space.
Definition 31 In a soft quad topological space $\left(\widetilde{X}, \widetilde{T}, \widetilde{T}, \widetilde{T}, \widetilde{T}, E\right)$

1) $\overline{T} \cup \overline{T}$ is said to be soft regular $T_0$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists $\overline{T} \cup \overline{T}$ soft regular open set $(F, \overline{E})$ and to $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E})$ and $y \notin (G, \overline{E})$ or $y \in (G, \overline{E})$ and $x \notin (G, \overline{E})$ similarly, to $\overline{T} \cup \overline{T}$ is said to be soft regular $T_0$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists $\overline{T} \cup \overline{T}$ regular open set $(F, \overline{E})$ and $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E})$ and $y \notin (G, \overline{E})$ or $y \in (G, \overline{E})$ and $x \notin (G, \overline{E})$.

Soft quad topological spaces $\left(\widetilde{X}, \widetilde{T}, \widetilde{T}, \widetilde{T}, \widetilde{T}, \widetilde{E}\right)$ is said to be pair wise soft regular $T_0$ space if $\overline{T} \cup \overline{T}$ is soft regular $T_0$ space with respect to $\overline{T} \cup \overline{T}$ and $\overline{T} \cup \overline{T}$ and is soft regular $T_0$ space with respect to $\overline{T} \cup \overline{T}$.

2) $\overline{T} \cup \overline{T}$ is said to be soft regular $T_1$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists $\overline{T} \cup \overline{T}$ soft regular open set $(F, \overline{E})$ and to $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E})$ and $y \notin (G, \overline{E})$ and $y \in (G, \overline{E})$ and $x \notin (G, \overline{E})$. Similarly, $\overline{T} \cup \overline{T}$ is said to be soft regular $T_1$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of distinct points $x, y \in X$ such that $x \neq y$ there exists $\overline{T} \cup \overline{T}$ soft regular open set $(F, \overline{E})$ and to $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E})$ and $y \notin (G, \overline{E})$ and $y \in (G, \overline{E})$ and $x \notin (G, \overline{E})$.

Soft quad topological spaces $\left(\widetilde{X}, \widetilde{T}, \widetilde{T}, \widetilde{T}, \widetilde{T}, \widetilde{E}\right)$ is said to be pair wise soft regular $T_1$ space if $\overline{T} \cup \overline{T}$ is soft regular $T_1$ space with respect to $\overline{T} \cup \overline{T}$ and to $\overline{T} \cup \overline{T}$ is soft regular $T_1$ space with respect to $\overline{T} \cup \overline{T}$.

3) $\overline{T} \cup \overline{T}$ is said to be soft regular $T_2$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists a $\overline{T} \cup \overline{T}$ soft regular open set $(F, \overline{E})$ and a $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E})$ and $y \notin (G, \overline{E})$, $(F, \overline{E}) \cap (G, \overline{E}) = \emptyset$. Similarly, $\overline{T} \cup \overline{T}$ is said to be soft regular $T_2$ space with respect to $\overline{T} \cup \overline{T}$ if for each pair of points $x, y \in X$ such that $x \neq y$ there exists a $\overline{T} \cup \overline{T}$ soft regular open set $(F, \overline{E})$ and a $\overline{T} \cup \overline{T}$ soft regular open set $(G, \overline{E})$ such that $x \in (F, \overline{E}), y \in (G, \overline{E})$.
and \((F, E) \cap (G, E) = \phi\). The soft quad totopological space \((X, \hat{T}, \hat{T}, \hat{T}, \hat{T}, \hat{T}, \hat{E})\) is said to be pair wise soft regular \(T_2\) space if \(\hat{T} \cup \hat{T}\) is soft regular \(T_2\) space with respect to \(\hat{T} \cup \hat{T}\) and \(\hat{T} \cup \hat{T}\) is soft regular \(T_2\) space with respect to \(\hat{T} \cup \hat{T}\).

**Definition 3.2** In a soft quad topological space \((X, \hat{T}, \hat{T}, \hat{T}, \hat{T}, \hat{T}, \hat{E})\)

1) \(\frac{1}{3} \hat{T} \cup \hat{T}\) is said to be soft regular \(qT_3\) space with respect to \(\tau_3 \cup \tau_4\) if \(\hat{T} \cup \hat{T}\) is soft regular \(T_1\) space with respect to \(\hat{T} \cup \hat{T}\) and for each pair of points \(x, y \in X\) such that \(x \neq y\) there exists \(\hat{T} \cup \hat{T}\) soft regular closed set \((G, E)\) such that \(x \notin (G, E), a \frac{1}{3} \hat{T} \cup \hat{T}\) soft regular open set \((F_1, \hat{E})\) and \(\frac{1}{3} \hat{T} \cup \hat{T}\) soft regular open set \((F_2, \hat{E})\) such that \(x \in (F_1, \hat{E}), (G, E) \subseteq (F_2, \hat{E})\) and \((F_1, \hat{E}) \cap (F_2, \hat{E}) = \phi\). Similarly, \(\hat{T} \cup \hat{T}\) is said to be soft regular \(T_3\) space with respect to \(\frac{1}{2} \hat{T} \cup \hat{T}\) if \(\hat{T} \cup \hat{T}\) is soft regular \(T_1\) space with respect to \(\tau_1 \cup \tau_2\) and for each pair of points \(x, y \in X\) such that \(x \neq y\) there exists \(\tau_3 \cup \tau_4\) soft regular closed set \((G, E)\) such that \(x \notin (G, E), \tau_3 \cup \tau_4\) soft regular open set \((F_1, \hat{E})\) and \(\frac{1}{2} \hat{T} \cup \hat{T}\) soft regular open set \((F_2, \hat{E})\) such that \(x \in (F_1, \hat{E}), (G, E) \subseteq (F_2, \hat{E})\) and \((F_1, \hat{E}) \cap (F_2, \hat{E}) = \phi\).

2) \(\frac{1}{3} \hat{T} \cup \hat{T}\) is said to be soft regular \(T_4\) space with respect to \(\frac{1}{2} \hat{T} \cup \hat{T}\) if \(\hat{T} \cup \hat{T}\) is soft regular \(T_1\) space with respect to \(\frac{1}{2} \hat{T} \cup \hat{T}\), there exists \(\frac{1}{2} \hat{T} \cup \hat{T}\) soft regular closed set \((F_1, \hat{E})\) and \(\frac{1}{2} \hat{T} \cup \hat{T}\) soft regular closed set \((F_2, \hat{E})\) such that \((F_1, \hat{E}) \cap (F_2, \hat{E}) = \phi\). Also there exists \((F_3, \hat{E})\) and \((G_1, \hat{E})\) such that \((F_3, \hat{E})\) is soft \(\hat{T} \cup \hat{T}\) regular open set, \((G_1, \hat{E})\) is soft \(\hat{T} \cup \hat{T}\) regular open set such that \((F_1, \hat{E}) \subseteq (F_3, \hat{E}), (F_2, \hat{E}) \subseteq (G_1, \hat{E})\). Similarly, \(\hat{T} \cup \hat{T}\) is said to be soft regular \(T_4\) space with respect to \(\frac{1}{2} \hat{T} \cup \hat{T}\) if \(\hat{T} \cup \hat{T}\) is soft regular \(T_1\) space with respect to \(\frac{3}{4} \hat{T} \cup \hat{T}\), there exists \(\frac{3}{4} \hat{T} \cup \hat{T}\) soft regular closed set \((F_1, \hat{E})\) and \(\frac{1}{2} \hat{T} \cup \hat{T}\) soft regular closed set \((F_2, \hat{E})\) such that \((F_1, \hat{E}) \cap (F_2, \hat{E}) = \phi\). Also there exist \((F_3, \hat{E})\) and \((G_1, \hat{E})\) such that \((F_3, \hat{E})\) is soft \(\frac{3}{4} \hat{T} \cup \hat{T}\) regular open set, \((G_1, \hat{E})\) is soft \(\frac{3}{4} \hat{T} \cup \hat{T}\) regular open set such that \((F_1, \hat{E}) \subseteq (F_3, \hat{E})\).
Let \((X, \tau, \bar{E})\) be a soft topological space over \(X\). If \((X, \tau, \bar{E})\) is soft-
\(R_3\)-space, then for all \(x \in \bar{X}\), \(x_\bar{E} = (x, \bar{E})\) is regular-closed soft set.

**Proof.** We want to prove that \(x_\bar{E}\) is regular-closed soft set, which is sufficient to prove
that \(x_\bar{E}\) is regular soft-open set for all \(y \in [x]^c\). Since \((X, \tau, \bar{E})\) is soft-
\(R_3\)-space, then there exists soft regular open sets \((F, \bar{E})_Y\) and \((G, \bar{E})\) such that \(y_\bar{E} \subseteq (F, \bar{E})_Y\) and \(x_\bar{E} \cap (F, \bar{E})_Y = \emptyset\) and \(x_\bar{E} \subseteq (G, \bar{E})\) and \(y_\bar{E} \cap (G, \bar{E}) = \emptyset\). It follows that \(U_{y \in (x)^c} (F, \bar{E})_y \subseteq x_\bar{E}\). Now, we want to prove that \(x_\bar{E} \subseteq \bigcup_{y \in (x)^c} (F, \bar{E})_y\). Let \(U_{y \in (x)^c} (F, \bar{E})_y = (H, \bar{E})\), where \(H(e) = U_{y \in (x)^c} (F, \bar{E})_y\) for all \(e \in \bar{E}\). Since \(x_\bar{E}(e) = [x]^c\) for all \(e \in \bar{E}\) from Definition 6, so, for all \(y \in [x]^c\) and \(e \in \bar{E}\), \(x_\bar{E}(e) = [x]^c = U_{y \in (x)^c} \{y\} = U_{y \in (x)^c} (F, \bar{E})_y\). Thus, \(x_\bar{E} \subseteq \bigcup_{y \in (x)^c} (F, \bar{E})_y\) from Definition 2, and \(so x_\bar{E} = \bigcup_{y \in (x)^c} (F, \bar{E})_y\). This means that \(x_\bar{E}\) is soft regular-open set for all \(y \in [x]^c\). Hence \(x_\bar{E}\) is soft regular-closed set.

**Proposition 2.** Let \((\bar{Y}, \bar{\tau}, \bar{E})\) be a soft sub space of a soft topological space \((\bar{X}, \tau, \bar{E})\)
and \((F, \bar{E}) \in SS(\bar{X})\) then,

1. If \((F, \bar{E})\) is soft regular open soft set in \(\bar{Y}\) and \(Y \in \tau\), then \((F, \bar{E}) \in \tau\).

2. \((F, \bar{E})\) is soft regular open soft set in \(\bar{Y}\) if and only if \((F, \bar{E}) = Y \cap (G, \bar{E})\) for some \((G, \bar{E}) \in \tau\).

3. \((F, \bar{E})\) is soft regular closed soft set in \(\bar{Y}\) if and only if \((F, \bar{E}) = \bar{Y} \cap (H, \bar{E})\) for some \((H, \bar{E})\) is \(\tau\) soft regular close set.

**Proof.** 1) Let \((F, \bar{E})\) be a soft regular open set in \(\bar{Y}\), then there does exists a soft regular open set \((G, \bar{E})\) in \(X\) such that \((F, \bar{E}) = Y \cap (G, \bar{E})\). Now, if \(Y \in \tau\) then \(Y \cap (G, \bar{E}) \in \tau\) by the third condition of the definition of a soft topological space and hence \((F, \bar{E}) \in \tau\).

2) Follows from the definition of a soft subspace.

3) If \((F, \bar{E})\) is soft regular closed in \(\bar{Y}\) then we have \((F, \bar{E}) = \bar{Y} \setminus (G, \bar{E})\), for some \((G, \bar{E}) \in \tau\). Now, \((G, \bar{E}) = Y \cap (H, \bar{E})\) for some soft regular open set \((H, \bar{E}) \in \tau\). for any \(\alpha \in \bar{E}\). \(F(\alpha) = Y \setminus G(\alpha) = Y \setminus (Y(\alpha) \cap H(\alpha)) = \bar{Y} \setminus (Y \cap (X \setminus H(\alpha))) = \bar{Y} \cap (H(\alpha))^c = \bar{Y} \cap (H(\alpha))^c\). Thus \((F, \bar{E}) = \bar{Y} \cap (H, \bar{E})\) is soft regular closed in \(\bar{X}\) as \((H, \bar{E}) \in \tau\). Conversely, suppose that \((F, \bar{E}) = \bar{Y} \cap (H, \bar{E})\)
(G, E) for some soft regular closed set (G, E) in X. This qualifies us to say that
(G, E) / ∈ τ. Now, if (G, E) = (X, E) \ (H, E) where (H, E) is soft regular open.

Proposition 3. Let \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) be a soft quad topological space over \( \hat{X} \). Then, if \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) and \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) are soft regular T_3 space then \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a pair wise soft regular T_2 space.

**Proof.** Suppose \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a soft regular T_3 space with respect to \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) then according to definition for \( x, y \in \hat{X} \), which distinct, by using Proposition 1, \( (\hat{Y}, \hat{E}) \) is soft regular closed set in \( \hat{\tau} \cup \hat{\tau} \) and \( x \notin (\hat{Y}, \hat{E}) \) there exists a \( \hat{\tau} \cup \hat{\tau} \) soft regular open set \( (F, E) \) and a \( \hat{\tau} \cup \hat{\tau} \) soft regular open set \( (G, E) \) such that \( x \in (F, E), y \in (G, E) \subseteq (G, E) \) and \( (F_1, E) \cap (F_2, E) = \phi \). Hence \( \hat{\tau} \cup \hat{\tau} \) is soft regular T_2 space with respect to \( \hat{\tau} \cup \hat{\tau} \). Similarly, if \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a soft regular T_3 space with respect to \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) then according to definition for \( x, y \in \hat{X}, x \neq y \), by using Theorem 2, \( (x, E) \) is regular closed soft set in \( \hat{\tau} \cup \hat{\tau} \) and \( y \notin (x, E) \) there exists a \( \hat{\tau} \cup \hat{\tau} \) soft regular open set \( (F, E) \) and a \( \hat{\tau} \cup \hat{\tau} \) soft regular open set \( (G, E) \) such that \( y \in (F, E), x \in (G, E) \subseteq (G, E) \) and \( (F_1, E) \cap (F_2, E) = \phi \). Hence \( \hat{\tau} \cup \hat{\tau} \) is soft regular T_2 space. This implies that \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a pair wise soft regular T_2 space.

Proposition 4. Let \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) be a soft quad topological space over \( \hat{X} \). \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) and \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) are soft regular T_3 space then \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a pair wise soft regular T_2 space.

**Proof.** Suppose \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) is a soft regular T_3 space with respect to \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) then according to definition for \( x, y \in X, x \neq y \) there exists a \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) soft regular open set \( (F, E) \) and a \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) soft regular open set \( (G, E) \) such that \( x \in (F, E) \) and \( y \notin (F, E) \) or \( y \in (G, E) \) and \( x \notin (G, E) \) and for each point \( x \in X \) and each \( \left( \hat{X}, \hat{\tau}, \hat{T}, \hat{\tau}, \hat{E} \right) \) regular closed soft set \( (G_1, E) \) such that \( x \notin (G_1, E) \) there exists
\( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F_1, E)\) and \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F_2, E)\) such that \( x \in (F_1, \tilde{E}), (G_1, \tilde{E}) \subseteq (F_2, \tilde{E}) \) and \((F_1, \tilde{E}) \cap (F_2, \tilde{E}) = \phi \). Similarly, to \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) is a soft regular \( T_3 \) space with respect to \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \). So according to definition for \( x, y \in X, x \neq y \) there exists a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F, \tilde{E})\) and a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((G, \tilde{E})\) such that \( x \in (F, \tilde{E}) \) and \( y \notin (F, \tilde{E}) \) or \( x \notin (G, \tilde{E}) \) and each point \( x \in \tilde{x} \) and each \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) regular closed soft set \((G_1, \tilde{E})\) such that \( x \notin (G_1, \tilde{E}) \) there exists \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F_1, \tilde{E})\) and a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F_2, \tilde{E})\) such that \( x \in (F_1, \tilde{E}), (G_1, \tilde{E}) \subseteq (F_2, \tilde{E}) \) and \((F_1, \tilde{E}) \cap (F_2, \tilde{E}) = \phi \). Hence \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) is pair wise soft regular \( T_3 \) space.

**Proposition 5.** If \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) be a soft quad topological space over \( X \). \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) and \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) are soft regular \( T_4 \) space then \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) is pair wise soft regular \( T_4 \) space.

**Proof.** Suppose \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) is soft regular \( T_4 \) space with respect to \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \). So according to definition for \( x, y \in \tilde{X}, x \neq y \) there exist a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((F, \tilde{E})\) and a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) soft regular open set \((G, \tilde{E})\) such that \( x \in (F, \tilde{E}) \) and \( y \notin (F, \tilde{E}) \) or \( y \in (G, \tilde{E}) \) and \( x \notin (G, \tilde{E}) \) each \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) soft regular closed set \((F_1, \tilde{E})\) and a \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) soft regular closed set \((F_2, \tilde{E})\) such that \( (F_1, \tilde{E}) \cap (F_2, \tilde{E}) = \phi \). There exist \((F_3, \tilde{E})\) and \((G_1, \tilde{E})\) such that \((F_3, \tilde{E})\) is soft \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) and soft regular open set \((G_1, \tilde{E})\) is soft \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) regular open set \((F_1, \tilde{E}) \subseteq (F_3, \tilde{E}), (F_2, \tilde{E}) \subseteq (G_1, \tilde{E}) \) and \((F_3, \tilde{E}) \cap (G_1, \tilde{E}) = \phi \). Similarly, \( \left( \tilde{X}, \tilde{T}, \tilde{\tau}, \tilde{\tau}, \tilde{E} \right) \) is soft regular \( T_4 \) space with respect to \( \tau_1 \) so according to definition for \( x, y \in \tilde{X}, x \neq y \) there
exists a \((X, \tau_3, \tau_4, E)\) soft regular open set \((F, E)\) and a \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) soft regular open set \((G, E)\) such that \(x \in (F, E)\) and \(y \notin (F, E)\) or \(y \in (G, E)\) and \(x \notin (G, E)\) and for each \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) soft regular closed set \((F_1, E)\) and \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) soft regular closed set \((F_2, E)\) such that \((F_1, E) \cap (F_2, E) = \emptyset\). there exists soft regular open sets\((F_3, E)\) and \((G_1, E)\) such that \((F_3, E)\) is soft \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) regular open set \((G_1, E)\) is soft \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) regular open set such that \((F_1, E) \subseteq (F_3, E), (F_2, E) \subseteq (G_1, E)\) and \((F_3, E) \cap (G_1, E) = \emptyset\). Hence \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_4\) space.

**Proposition 6.** Let \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) be a soft quad topological space over \(X\) and \(Y\) be a non-empty subset of \(X\). if \(\left(\bar{Y}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_3\) space. Then \(\left(\bar{Y}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_3\) space.

**Proof.** First we prove that \(\left(\bar{Y}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_1\) space.

Let \(x, y \in X, x \neq y\) if \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise space then this implies that \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft space. So there exists \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) soft regular open \((F, E)\) and \(\left(\bar{X}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) soft regular open set \((G, E)\) such that \(x \in (F, E)\) and \(y \notin (F, E)\) or \(y \in (G, E)\) and \(x \notin (G, E)\) now \(x \in \bar{Y}\) and \(x \notin (G, E)\). Hence \(x \in \bar{Y} \cap (F, E) = (Y, F, E)\) then \(y \notin Y \cap (\alpha)\) for some \(\alpha \in E\). this means that \(\alpha \in \bar{E}\) then \(y \notin Y \cap F (\alpha)\) for some \(\alpha \in \bar{E}\).

Therefore, \(y \notin Y \cap (F, E) = (Y, E)\). Now \(y \in \bar{Y}\) and \(y \notin (G, E)\). Hence \(y \in Y \cap (G, E) = (G, E)\) where \(G, E) \in (X, \tau_3, \tau_4, E)\). Consider \(x \notin (G, E)\) this means that \(\alpha \in \bar{E}\) then \(x \notin Y \cap G (\alpha)\) for some \(\alpha \in \bar{E}\). Therefore \(x \notin \bar{Y} \cap (G, E) = (G, E)\) thus \(\left(\bar{Y}, \tau_1, \tau_2, \tau_3, \tau_4, \tau, \bar{E}\right)\) is pair wise soft regular \(T_1\) space.

Now we prove that \(\left(\bar{Y}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_3\) space then \(\left(\bar{Y}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{\tau}, \bar{E}\right)\) is pair wise soft regular \(T_3\) space.
Let \( y \in \bar{Y} \) and \((G, \bar{E})\) be a soft regular closed set in \( Y \) such that \( y \notin (G, \bar{E}) \) where \((G, \bar{E}) \in \left\{ \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right\}\) then \((G, \bar{E}) = (Y, \bar{E}) \cap (F, \bar{E})\) for some soft regular closed set in \((\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E})\) Hence \( y \notin (Y, \bar{E}) \cap (F, \bar{E}) \) but \( y \in (Y, \bar{E}) \), so \( y \notin (F, \bar{E}) \) since \((\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E})\) is soft regular \( T_3 \) space \((X, \tau_1, \tau_2, \tau_3, \tau_4, \bar{E})\) is soft regular space so there exists \((\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{E})\) soft regular open set \((F_1, \bar{E})\) and \((\bar{X}, \bar{T}, \bar{T}, \bar{E})\) soft regular open set \((F_2, \bar{E})\) such that

\[
y \in (F_1, \bar{E}), (G, \bar{E}) \subseteq (F_2, \bar{E})\]

\[
(F_1, \bar{E}) \cap (F_2, \bar{E}) = \phi
\]

Take \((G_1, \bar{E}) = (Y, \bar{E}) \cap (F_2, \bar{E})\) then \((G_1, \bar{E}), (G_2, \bar{E})\) are soft regular open set in \( \bar{Y} \) such that

\[
y \in (G_1, \bar{E}), (G, \bar{E}) \subseteq (Y, \bar{E}) \cap (F_2, \bar{E}) = (G_2, \bar{E})\]

\[
(G_1, \bar{E}) \cap (G_2, \bar{E}) = (G_1, \bar{E}) \cap (F_1, \bar{E}) \cap (F_2, \bar{E}) = \phi
\]

\[
(G_1, \bar{E}) \cap (G_2, \bar{E}) = \phi
\]

There fore \( \bar{T} \cup \bar{T} \) is soft R-regular space with respect to \( \bar{T} \cup \bar{T} \). Similarly, Let \( y \in \bar{Y} \) and \((G, \bar{E})\) be a soft regular closed sub set in \( \bar{Y} \) such that \( y \notin (G, \bar{E}) \), where \((G, \bar{E}) \in \left\{ \bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E} \right\}\) then \((G, \bar{E}) = (Y, \bar{E}) \cap (F, \bar{E})\) where \((F, \bar{E})\) is some soft regular closed set in \((\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{E})\) \( y \notin (Y, \bar{E}) \cap (F, \bar{E}) \) But \( y \in (Y, \bar{E}) \) so \( y \notin (F, \bar{E}) \) since \((\bar{X}, \bar{T}, \bar{T}, \bar{E})\) is soft R-regular space so there exists \((\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{E})\) soft regular open set \((F_1, \bar{E})\) and \((\bar{X}, \bar{T}, \bar{T}, \bar{E})\) soft regular open set \((F_2, \bar{E})\). Such that

\[
y \in (F_1, \bar{E}), (G, \bar{E}) \subseteq (F_2, \bar{E})\]

\[
(F_1, \bar{E}) \cap (F_2, \bar{E}) = \phi
\]

Take

\[
(G_1, \bar{E}) = (Y, \bar{E}) \cap (F_1, \bar{E})\]

\[
(G_1, \bar{E}) = (Y, \bar{E}) \cap (F_1, \bar{E})\]

Then \((G_1, \bar{E})\) and \((G_2, \bar{E})\) are soft regular open set in \( Y \) such that

\[
y \in (G_1, \bar{E}), (G, \bar{E}) \subseteq (Y, \bar{E}) \cap (F_2, \bar{E}) = (G_2, \bar{E})\]

\[
(G_1, \bar{E}) \cap (G_2, \bar{E}) \subseteq (F_1, \bar{E}) \cap (F_2, \bar{E}) = \phi
\]
Therefore $\mathcal{T} \cup \mathcal{U}$ is soft regular space with respect $\mathcal{T} \cup \mathcal{U} \Rightarrow (\bar{Y}, \mathcal{T}, \mathcal{T}, \mathcal{E})$ is pair wise soft regular $T_3$ space.

**Proposition 7.** Let $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ be a soft quad topological space over $\bar{X}$ and $\bar{Y}$ be a soft regular closed sub space of $X$. If $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ is pair wise soft regular $T_4$ space then $(\bar{Y}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E})$ is pair wise soft regular $T_4$ space.

**Proof.** Since $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ is pair wise soft regular $T_4$ space. So this implies that $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ is pair wise soft regular $T_1$ space as proved above.

We prove $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ is pair wise soft regular normal space. Let $(G_1, \mathcal{E}), (G_2, \mathcal{E})$ be soft regular closed sets in $\bar{Y}$ such that

$$(G_1, \mathcal{E}) \cap (G_2, \mathcal{E}) = \emptyset$$

Then

$$(G_1, \mathcal{E}) = (Y, \mathcal{E}) \cap (F_1, \mathcal{E})$$

and

$$(G_2, \mathcal{E}) = (Y, \mathcal{E}) \cap (F_2, \mathcal{E})$$

For some soft regular closed sets such that $(F_1, \mathcal{E})$ is soft regular closed set in $\mathcal{T} \cup \mathcal{T}$ soft regular closed set $(F_2, \mathcal{E})$ in $\mathcal{T} \cup \mathcal{T}$ and $(F_1, \mathcal{E}) \cap (F_2, \mathcal{E}) = \emptyset$ From Proposition 2. Since, $\bar{Y}$ is soft regular closed sub set of $X$ then $(G_1, \mathcal{E}), (G_2, \mathcal{E})$ are soft regular closed sets in $\bar{X}$ such that

$$(G_1, \mathcal{E}) \cap (G_2, \mathcal{E}) = \emptyset$$

Since $\left(\bar{X}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{T}, \mathcal{E}\right)$ is pair wise softs regular normal space. So there exists soft regular open sets $(H_1, \mathcal{E})$ and $(H_2, \mathcal{E})$ such that $(H_1, \mathcal{E})$ is soft regular open set in $\mathcal{T}_1 \cup \mathcal{T}_2$ and $(H_2, \mathcal{E})$ is soft regular open set in $\mathcal{T}_1 \cup \mathcal{T}_2$ such that

$$(G_1, \mathcal{E}) \subseteq (H_1, \mathcal{E})$$

$$(G_2, \mathcal{E}) \subseteq (H_2, \mathcal{E})$$

$$(H_1, \mathcal{E}) \cap (H_2, \mathcal{E}) = \emptyset$$
Since 
\[(G_1, \bar{E}), (G_2, \bar{E}) \subseteq (Y, E)\]
Then
\[(G_1, \bar{E}) \subseteq (Y, \bar{E}) \cap (H_1, \bar{E})\]
\[(G_2, \bar{E}) \subseteq (Y, \bar{E}) \cap (H_2, \bar{E})\]
and
\[
[(Y, \bar{E}) \cap (H_1, \bar{E})] \cap [(Y, \bar{E}) \cap (H_2, \bar{E})] = \phi
\]

Where \((Y, \bar{E}) \cap (H_1, \bar{E})\) and \((Y, \bar{E}) \cap (H_2, \bar{E})\) are soft regular open sets in \(Y\) there fore \(1^Y \cup 2^Y\) \(\bar{T} \cup \bar{T}\) is soft regular normal space with respect to \(\bar{T} \cup \bar{T}\). Similarly, Let \((G_1, \bar{E}), (G_2, \bar{E})\) be soft regular closed sub set in \(Y\) such that
\[
(G_1, \bar{E}) \cap (G_2, \bar{E}) = \phi
\]
Then
\[
(G_1, \bar{E}) = (Y, \bar{E}) \cap (F_1, \bar{E})
\]
and
\[
(G_2, \bar{E}) = (Y, \bar{E}) \cap (F_2, \bar{E})
\]

For some soft regular closed sets such that \((F_1, \bar{E})\) is soft regular closed set in \(1^Y \cup 2^Y\) \(\bar{T} \cup \bar{T}\) and \((F_2, \bar{E})\) soft regular closed set in \(3^Y \cup 4^Y\) \(\bar{T} \cup \bar{T}\) and \((F_1, \bar{E})(F_2, \bar{E}) = \phi\) from Proposition 2. Since, \(Y\) is soft regular closed sub set in \(X\) then \((G_1, \bar{E}), (G_2, \bar{E})\) are soft regular closed sets in \(X\) such that
\[
(G_1, \bar{E}) \cap (G_2, \bar{E}) = \phi
\]

Since \(\bar{X}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{E}\) is pair wise soft regular normal space so there exists soft regular open sets \((H_1, \bar{E})\) and \((H_2, \bar{E})\). Such that \((H_1, \bar{E})\) is soft regular open set is \(1^Y \cup 2^Y\) \(\bar{T} \cup \bar{T}\) and \((H_2, \bar{E})\) is soft regular open set in \(\bar{T} \cup \bar{T}\) such that
\[
(G_1, \bar{E}) \subseteq (H_1, \bar{E})
\]
\[
(G_2, \bar{E}) \subseteq (H_2, \bar{E})
\]
\[
(H_1, \bar{E}) \cap (H_2, \bar{E}) = \phi
\]

Since
\[
(G_1, \bar{E}), (G_2, \bar{E}) \subseteq (Y, E)
\]
Then
\[
(G_1, \bar{E}) \subseteq (Y, \bar{E}) \cap (H_1, \bar{E})
\]
\[
(G_2, \bar{E}) \subseteq (Y, \bar{E}) \cap (H_2, \bar{E})
\]
and
\[
[(Y, \bar{E}) \cap (H_1, \bar{E})] \cap [(Y, \bar{E}) \cap (H_2, \bar{E})] = \phi
\]
Where \((Y, \overline{E}) \cap (H_1, \overline{E})\) and \((Y, \overline{E}) \cap (H_2, \overline{E})\) are soft regular open sets in \(\overline{Y}\) there fore \(\overline{Y} \cup \overline{Y}\) is soft regular normal space with respect to \(\overline{Y} \cup \overline{Y}\) \(\Rightarrow (\overline{Y}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{E})\) is pair wise soft regular \(T_4\) space.

### 3.2 Soft Regular Separation Axioms in Soft Quad Topological Spaces with Respect to Soft Points

In this section, we introduced soft topological structures known as soft regular separation axioms in soft quad topology with respect to soft points. With the applications of these soft regular separation axioms different result are brought under examination.

**Definition 33** In a soft quad topological space \((\overline{X}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{E})\)

1) \(\overline{1} \overline{2} \overline{T} \cup \overline{T}\) said to be soft regular \(T_0\) space with respect to \(\overline{3} \overline{4} \overline{T} \cup \overline{T}\) if for each pair of distinct points \(e_G, e_H \in X_A\) there happens \(\overline{T} \cup \overline{T}\) soft regular open set \((F, \overline{E})\) and a \(\overline{1} \overline{2} \overline{T}\) soft regular open set \((G, \overline{E})\) such that \(e_G \in (F, \overline{E})\) and \(e_H \notin (G, \overline{E})\). Similarly, \(\overline{3} \overline{4} \overline{T}\) is said to be soft regular \(T_0\) space with respect to \(\overline{1} \overline{2} \overline{T} \cup \overline{T}\) if for each pair of distinct points \(e_G, e_H \in X_A\) there happens \(\overline{T} \cup \overline{T}\) soft regular open set \((F, \overline{E})\) and a \(\overline{3} \overline{4} \overline{T}\) soft regular open set \((G, \overline{E})\) such that \(e_G \notin (F, \overline{E})\) and \(e_H \in (G, \overline{E})\) and \(e_G \notin (G, \overline{E})\). Soft quad topological spaces \((\overline{X}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{E})\) is said to be pair wise soft regular \(T_0\) space if \(\overline{1} \overline{2} \overline{T} \cup \overline{T}\) is soft regular \(T_0\) space with respect to \(\overline{3} \overline{4} \overline{T} \cup \overline{T}\) and \(\overline{3} \overline{4} \overline{T}\) is soft regular \(T_0\) spaces with respect to \(\overline{1} \overline{2} \overline{T} \cup \overline{T}\).

2) \(\overline{1} \overline{2} \overline{T} \cup \overline{T}\) is said to be soft regular \(T_1\) space with respect to \(\overline{3} \overline{4} \overline{T} \cup \overline{T}\) if for each pair of distinct points \(e_G, e_H \in X_A\) there exists \(\overline{T} \cup \overline{T}\) soft regular open set \((F, \overline{E})\) and \(\overline{3} \overline{4} \overline{T}\) soft regular open set \((G, \overline{E})\) such that \(e_G \in (F, \overline{E})\) and \(e_H \notin (G, \overline{E})\) and \(e_G \notin (G, \overline{E})\). Similarly, \(\overline{1} \overline{2} \overline{T}\) is said to be soft regular \(T_1\) space with respect to \(\overline{3} \overline{4} \overline{T}\) if for each pair of distinct points \(e_G, e_H \in X_A\) there exist \(\overline{T} \cup \overline{T}\) soft regular open set \((F, \overline{E})\) and a \(\overline{1} \overline{2} \overline{T}\) soft regular open set \((G, \overline{E})\) such that \(e_G \in (F, \overline{E})\) and \(e_H \notin (G, \overline{E})\) and \(e_G \notin (G, \overline{E})\). Soft quad topological space \((\overline{X}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{T}, \overline{E})\) is said to be pair wise soft regular \(T_1\) space if \(\overline{3} \overline{4} \overline{T}\) is soft regular \(T_1\) space with respect to \(\overline{1} \overline{2} \overline{T}\) and \(\overline{3} \overline{4} \overline{T}\) is soft regular \(T_1\) spaces with respect to \(\overline{1} \overline{2} \overline{T}\).
3) \( \overline{\tau} \cup \overline{\tau} \) is said to be soft regular \( T_2 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \), if for each pair of distinct points \( e_G, e_H \in X_A \) there exists a \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F, \overline{E}) \) and a \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (G, \overline{E}) \) such that \( e_G \in (F, \overline{E}) \) and \( e_H \not\in (G, \overline{E}) \) and \( e_G \not\in (G, \overline{E}) \) and \( (F, \overline{E}) \cap (G, \overline{E}) = \emptyset \). Similarly, \( \tau_3 \cup \tau_4 \) is said to be soft regular \( T_2 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) if for each pair of distinct points \( e_G, e_H \in X_A \) there exists a \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F, \overline{E}) \) and a \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (G, \overline{E}) \) such that \( e_G \in (F, \overline{E}) \) and \( e_H \in (G, \overline{E}) \) and \( (F, \overline{E}) \cap (G, \overline{E}) = \emptyset \). The soft quad topological space \( (\overline{X}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{E}) \) is said to be pair wise soft regular \( T_2 \) space if \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_2 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) and \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_2 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \).

**Definition 34** In a soft quad topological space \( (\overline{X}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{E}) \)

1) \( \overline{\tau} \cup \overline{\tau} \) is said to be soft regular \( T_3 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) if \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_1 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) and for each pair of distinct points \( e_G, e_H \in X_A \) there exists \( \overline{\tau} \cup \overline{\tau} \).

regular closed soft set \( (G, \overline{E}) \) such that \( e_G \not\in (G, \overline{E}), \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F_1, \overline{E}) \) and \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F_2, \overline{E}) \) such that \( e_G \in (F_1, \overline{E}), (G, \overline{E}) \subseteq (F_2, \overline{E}) \) and \( (F_1, \overline{E}) \cap (F_2, \overline{E}) = \emptyset \). Similarly, \( \tau_3 \cup \tau_4 \) is said to be soft regular \( T_3 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) if \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_3 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) and for each pair of distinct points \( e_G, e_H \in X_A \) there exists a \( \overline{\tau} \cup \overline{\tau} \) soft regular closed set \( (G, \overline{E}) \) such that \( e_G \not\in (G, \overline{E}), \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F_1, \overline{E}) \) and \( \overline{\tau} \cup \overline{\tau} \) soft regular open set \( (F_2, \overline{E}) \) such that \( e_G \in (F_1, \overline{E}), (G, \overline{E}) \subseteq (F_2, \overline{E}) \) and \( (F_1, \overline{E}) \cap (F_2, \overline{E}) = \emptyset \). The soft quad topological space \( (\overline{X}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{\tau}, \overline{E}) \) is said to be pair wise soft regular \( T_3 \) space if \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_3 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \).

2) \( \overline{\tau} \cup \overline{\tau} \) is said to be soft regular \( T_4 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \) if \( \overline{\tau} \cup \overline{\tau} \) is soft regular \( T_1 \) space with respect to \( \overline{\tau} \cup \overline{\tau} \), there exists \( \overline{\tau} \cup \overline{\tau} \) soft regular closed set \( (F_1, \overline{E}) \) and \( \overline{\tau} \cup \overline{\tau} \) soft regular closed set \( (F_2, \overline{E}) \) such that \( (F_1, \overline{E}) \cap (F_2, \overline{E}) = \emptyset \) and \( e_G, e_H \in X_A \) there exists \( (F_3, \overline{E}) \) and \( (G_1, \overline{E}) \) such that \( (F_3, \overline{E}) \) is soft \( \overline{\tau} \cup \overline{\tau} \) regular.
open set. $(G_1, E)$ is soft $\frac{3}{2} \cup \frac{4}{4}$ regular set such that $(F_1, E) \subseteq (F_3, E)$. Similarly, $\frac{3}{2} \cup \frac{4}{4}$ is said to be soft regular $T_4$ space with respect to $\frac{3}{2} \cup \frac{4}{4}$ if $\frac{3}{2} \cup \frac{4}{4}$ is soft regular $T_1$ space with respect to $\frac{3}{2} \cup \frac{4}{4}$ there exists $\frac{3}{2} \cup \frac{4}{4}$ soft regular closed set $(F_1, E)$ and $\frac{3}{2} \cup \frac{4}{4}$ soft regular closed set $(F_2, E)$ such that $(F_1, E) \cap (F_2, E) = \phi$. Also there exists $(F_3, E)$ and $(G_1, E)$ such that $(F_3, E)$ is soft $\frac{3}{2} \cup \frac{4}{4}$ regular open set, $(G_1, E)$ is soft $\frac{3}{2} \cup \frac{4}{4}$ regular soft set such that $(F_1, E) \subseteq (F_2, E) \subseteq (G_1, E)$ and $(F_3, E) \cap (G_1, E) = \phi$. Thus, $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ is said to be pair wise soft regular $T_4$ space if $\frac{3}{2} \cup \frac{4}{4}$ is soft regular $T_4$ space with respect to $\frac{3}{2} \cup \frac{4}{4}$ and to $\frac{3}{2} \cup \frac{4}{4}$ is soft regular $T_4$ space with respect to $\frac{3}{2} \cup \frac{4}{4}$.

**Proposition 8.** Let $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ be a soft topological space over $X$. $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ is soft regular $T_3$ space, then for all $e_G \in X_G$ $e_G = (e_G, E)$ is soft regular closed set.

**Proof.** We want to prove that $e_G$ is regular closed soft set, which is sufficient to prove that $e_G^C$ is regular open soft set for all $e_H \in \{e_G\}^C$. Since $(X, \tau_1, \tau_2, \tau_3, \tau_4, E)$ is soft regular $T_3$ space, then there exists soft regular sets $(F, E)_{e_H}$ and $(G, E)$ such that $e_{H_E} \subseteq (F, E)_{e_H}$ and $e_{G_E} \cap (F, E)_{e_H} = \phi$ and $e_{G_E} \subseteq (G, E)$ and $e_{H_E} \cap (G, E) = \phi$. It follows that, $\cup_{e_H \in (e_G)^C} (F, E)_{e_H} = (H, E)$. Where $H (e) = \cup_{e_H \in (e_G)^C} (F, E)_{e_H}$ for all $e \in E$. Since $e_{G_E}^C (e) = (e_G)^C$ for all $e \in E$ from Definition 9, so, for all $e_H \in (e_G)^C$ and $e \in E$ $e_{G_E}^C (e) = \{e_G\}^C = \cup_{e_H \in (e_G)^C} \{e_G\}^C = \cup_{e_H \in (e_G)^C} (F, E)_{e_H} = (H, E)$. Thus, $e_{G_E}^C \cup_{e_H \in (e_G)^C} (F, E)_{e_H}$ from Definition 2, and so, $e_{G_E}^C = \cup_{e_H \in (e_G)^C} (F, E)_{e_H}$. This means that, $e_{G_E}^C$ is soft regular open set for all $e_H \in (e_G)^C$. Therefore, $e_{G_E}$ is regular closed soft set.

**Proposition 9.** Let $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ be a soft quad topological space over $X$. Then, if $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ and $\left(\frac{1}{3}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ are soft regular $qT_3$ space, then $\left(\frac{1}{1}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E, E\right)$ is a pair wise soft regular $T_2$ space.

**Proof.** Suppose if $\left(\frac{1}{1}, \frac{2}{2}, \frac{3}{4}, \frac{4}{4}, E\right)$ is a soft regular $T_3$ space with respect to $\left(\frac{3}{4}, \frac{4}{4}, E, E\right)$, then according to definition for, $e_G \neq e_H, e_G, e_H \in X_A$, by using Theorem 8, $(e_H, E)$ is
soft regular closed set in \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) and \(e_G \notin (e_H, E)\) there exist a \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) soft regular open set \((F, E)\) and a \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) soft regular open set \((G, E)\) such that \(e_G \in (F, E)\), \(e_H \in (y, E) \subseteq (G, E)\) and \((F_1, E) \cap (F_2, E) = \phi\). Hence \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is soft regular \(T_2\) space with respect to \((\tilde{X}, \tilde{T}, \tilde{T}, E)\). Similarly, if \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a soft regular \(T_3\) space with respect to \((\tilde{X}, \tilde{T}, \tilde{T}, E)\), then according to definition for \(e_G \neq e_H, e_G, e_H \in X_A\), by using Theorem 8, \((e_G, \tilde{E})\) is regular closed soft set in \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) and \(y \notin (x, \tilde{E})\) there exists a \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) soft regular open set \((F, \tilde{E})\) and a \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) soft regular open set \((G, \tilde{E})\) such that \(e_H \in (F, \tilde{E})\), \(e_G \in (x, \tilde{E}) \subseteq (G, \tilde{E})\) and \((F_1, \tilde{E}) \cap (F_2, \tilde{E}) = \phi\). Hence \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a soft regular \(T_2\) space. Thus \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a pair wise soft regular \(T_2\) space.

**Proposition 10.** Let \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) be a soft quad topological space over \(\tilde{X}\). if \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) and \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) are soft regular \(T_3\) space then \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a pair wise soft regular \(T_3\) space.

**Proof.** Suppose \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a soft regular \(T_3\) space with respect to \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) then according to definition for \(e_G, e_H \in X_A e_G \neq e_H\) there happens \(\tau_1 \cup \tau_2\) soft regular open set \((F, E)\) and a \(\tilde{T} \cup \tilde{T}\) soft regular open set \((G, E)\) such that \(e_G \in (F, \tilde{E})\) and \(e_H \notin (F, E)\) or \(e_H \in (G, \tilde{E})\) and \(e_G \notin (G, E)\) and for each point \(e_G \in X_A\) and each \(\tilde{T} \cup \tilde{T}\) regular closed soft set \((G_1, \tilde{E})\) such that \(e_G \notin (G_1, \tilde{E})\) there happens a \(\tilde{T} \cup \tilde{T}\) soft regular open set \((F_1, \tilde{E})\) and \(\tilde{T} \cup \tilde{T}\) soft regular open set \((F_2, \tilde{E})\) such that \(e_G \in (F_1, \tilde{E}), (G_1, \tilde{E}) \subseteq (F_2, \tilde{E})\) and \((F_1, \tilde{E}) \cap (F_2, \tilde{E}) = \phi\). Similarly \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) is a soft regular \(T_3\) space with respect to \((\tilde{X}, \tilde{T}, \tilde{T}, E)\) So according to definition for \(e_G, e_H \in X_A\), \(e_G \neq e_H\) there exists a \(\tilde{T} \cup \tilde{T}\) soft regular open set \((F, \tilde{E})\) and \(\tilde{T} \cup \tilde{T}\) soft regular open set \((G, \tilde{E})\) such that \(e_H \in (F, \tilde{E})\) and \(e_H \notin (F, \tilde{E})\) or \(e_H \in (G, \tilde{E})\) and
Proposition 11. if \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) be a soft quad topological space over \(\mathcal{Z}\). \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) and \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) are soft regular space then \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) is pair wise soft regular space.

Proof. Suppose \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) is soft regular \(\mathcal{T}_4\) space with respect to \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\). So according to definition for \(e_G, e_H \in \mathcal{X}, e_G \neq e_H\) there happens a \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular open set \((F, \mathcal{E})\) and a \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular open set \((G, \mathcal{E})\) such that \(e_G \in (F, \mathcal{E})\) and \(e_H \notin (F, \mathcal{E})\) or \(e_H \in (G, \mathcal{E})\) and \(e_G \notin (G, \mathcal{E})\) each \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular closed set \((F_1, \mathcal{E})\) and a \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular closed set \((F_2, \mathcal{E})\) such that \((F_1, \mathcal{E}) \cap (F_2, \mathcal{E}) = \emptyset\). Similarly, \(\mathcal{T}_1 \cup \mathcal{T}_2\) is soft regular \(\mathcal{T}_4\) space with respect to \(\mathcal{T}_1 \cup \mathcal{T}_2\). So according to definition for \(e_G, e_H \in \mathcal{X}, e_G \neq e_H\) there happens a \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular open set \((F, \mathcal{E})\) and a \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular open set \((G, \mathcal{E})\) such that \(e_G \in (F, \mathcal{E})\) and \(e_H \notin (F, \mathcal{E})\) or \(e_H \in (G, \mathcal{E})\) and \(e_G \notin (G, \mathcal{E})\) for each \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular closed set \((F_1, \mathcal{E})\) and \(\mathcal{T}_1 \cup \mathcal{T}_2\) soft regular closed set \((F_2, \mathcal{E})\) such that \((F_1, \mathcal{E}) \cap (F_2, \mathcal{E}) = \emptyset\). Hence \(\mathcal{X}, \mathcal{T}, \mathcal{R}, \mathcal{E}\) is pair wise soft regular \(\mathcal{T}_3\) space.
Proposition 12. Let \( \{X, \tau, \bar{\tau}, \tilde{\tau}, E\} \) be a soft quad topological space over \( X \) and \( Y \) be a non-empty subset of \( X \). If \( (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) is pair wise soft regular \( T_3 \) space. Then \( (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) is pair wise soft regular \( T_3 \) space.

Proof. First we prove that \( (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) is pair wise soft regular space. Let \( O, P \in X \) if \( Y \) is pair wise soft space then this implies that \( Y \) is pair wise soft regular space. So there exists \( O \subseteq T \subseteq \bar{T} \) soft regular open set \((G, \bar{E})\) such that \( O \notin (G, \bar{E}) \) and \( P \notin (G, \bar{E}) \) for some \( \alpha \in \bar{E} \). Hence \( e_G \in Y \) and \( e_H \in (G, \bar{E}) \). Hence, \( e_H \in Y \) and \( e_H \in (G, \bar{E}) \). Consider \( x \notin (G, \bar{E}) \) this means that \( \alpha \in E \) then \( x \notin \bar{E} \) and \( x \notin \bar{E} \) for some \( \alpha \in \bar{E} \). Hence \( e_G \notin \bar{Y} \) and \( (G, \bar{E}) = (G, \bar{E}) \) thus \( (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) is pair wise soft regular \( T_1 \) space.

Now, we prove that \( (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) is pair wise soft regular \( T_3 \) space.

Let \( e_H \in \bar{Y} \) and \( (G, \bar{E}) \) be soft regular closed set in \( \bar{Y} \) such that \( e_H \notin (G, \bar{E}) \) where \( (G, \bar{E}) \subseteq \bar{T} \) then \( (G, \bar{E}) \subseteq (Y, \tau, \bar{\tau}, \tilde{\tau}, E) \) for some soft regular closed set in \( \bar{T} \). Hence \( e_H \notin (Y, \bar{E}) \) and \( e_H \notin (F, \bar{E}) \) for some \( \alpha \in \bar{E} \). But \( e_H \in (Y, \bar{E}) \), so \( e_H \notin (F, \bar{E}) \) since \( (X, \tau, \bar{\tau}, \tilde{\tau}, E) \) is soft regular \( T_3 \) space \( (X, \tau, \bar{\tau}, \tilde{\tau}, E) \) is soft R-regular space so there happens \( \tau_1 \cup \tau_2 \) soft regular open set \((F_1, \bar{E})\) and \( \tau_3 \cup \tau_4 \) soft regular open set \((F_2, \bar{E})\) such that

\[
e_H \in (F_1, \bar{E}), (G, \bar{E}) \subseteq (F_2, \bar{E}) \cup (F_1, \bar{E}) = \emptyset
\]

Take \( (G_1, E) = (Y, E) \cap (F_2, \bar{E}) \) then \( (G_1, E), (G_2, E) \) are soft regular open sets in \( Y \) such that

\[
e_H \in (G_1, \bar{E}), (G, \bar{E}) \subseteq (Y, \bar{E}) \cap (F_2, \bar{E}) = (G_2, \bar{E})
\]
Therefore, \( \left( \frac{1}{3}, \frac{2}{4}, \frac{3}{4} \right) \) is a soft R-regular space with respect to \( \bar{\mathcal{T}} \cup \mathcal{T} \). Similarly, let \( e_H \in \bar{\mathcal{Y}} \) and \( (G, \bar{E}) \) be a soft regular closed sub set in \( \bar{\mathcal{Y}} \) such that \( e_H \notin (G, \bar{E}) \). Where \( (G, E) \in \frac{3}{4} \mathcal{T} \cap \mathcal{T} \) then \( (G, \bar{E}) = (Y, \bar{E}) \cap (F, \bar{E}) \) where \( (F, \bar{E}) \) is some soft regular closed set in \( \bar{\mathcal{T}} \cup \mathcal{T} \). \( e_H \notin (Y, E) \) but \( e_H \notin (Y, E) \) so \( e_H \notin (F, E) \) since \( \left( X, \frac{1}{3}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \right) \) is a soft R-regular space so there happens \( \frac{3}{4} \mathcal{T} \cup \frac{4}{4} \mathcal{T} \) soft regular open set \( (F_1, \bar{E}) \) and \( \frac{1}{4} \mathcal{T} \cup \frac{2}{4} \mathcal{T} \) soft regular open set \( (F_2, \bar{E}) \). Such that

\[
\text{Take} \\ (G_1, \bar{E}) = (Y, \bar{E}) \cap (F_1, \bar{E}) \\
(G_1, \bar{E}) = (Y, \bar{E}) \cap (F_1, \bar{E})
\]

Then \( (G_1, \bar{E}) \) and \( (G_2, \bar{E}) \) are soft regular open set in \( \bar{\mathcal{Y}} \) such that

\[
\text{Take} \\ (G_1, \bar{E}), (G, \bar{E}) \subseteq (Y, \bar{E}) \cap (F_2, \bar{E}) = (G_2, \bar{E}) \\
(G_1, \bar{E}) \cap (G_2, \bar{E}) \subseteq (F_1, \bar{E}) \cap (F_2, \bar{E}) = \emptyset.
\]

Therefore, \( \frac{3}{4} \mathcal{T} \cup \frac{4}{4} \mathcal{T} \) is a soft R-regular space.

### 4 Conclusions

A soft set with single specific topological structure is unable to shoulder up the responsibility to construct the whole theory. So to make the theory healthy, some additional structures on soft set has to be introduced. It makes, it more springy to develop the soft topological spaces with its infinite applications. In this regards we introduced soft topological structure known as soft quad topological structure with respect to soft regular open sets. In this article, topology is the supreme branch of pure mathematics which deals with mathematical structures. Freshly, many scholars have studied the soft set theory which is coined by Molodtsov [4] and carefully applied to many difficulties which contain uncertainties in our social life. Shabir and Naz [7] familiarized and deeply studied the origin of soft topological spaces. They also studied topological structures and exhibited their several properties with respect to ordinary points.

In the present work, we constantly study the behavior of soft regular separation axioms in soft quad topological spaces with respect to soft points as well as ordinary points of a
soft topological space. We introduce soft regular $qT_0$ structure, soft regular $qT_1$ structure, soft regular $qT_2$ structure, soft regular $qT_3$ and soft regular $qT_4$ structure with respect to soft and ordinary points. In future we will plant these structures in different results. More over defined soft regular $T_0$ structure w. r. t. soft regular $T_1$ structure and vice versa, soft regular $T_2$ structure w. r. t soft regular $T_3$ and vice versa and soft regular $T_3$ space w. r. t soft regular $T_4$ and vice versa with respect to ordinary and soft points in soft quad topological spaces and studied their activities in different results with respect to ordinary and soft points. We also planted these axioms to different results. These soft semi separation axioms in quad structure would be valuable for the development of the theory of soft topology to solve complicated problems, comprising doubts in economics, engineering, medical etc. We also attractively discussed some soft transmissible properties with respect to ordinary as well as soft points. I have fastidiously studied numerous homes on the behalf of soft topology, and lastly I determined that soft topology is totally linked or in other sense we can correctly say that soft topology (Separation Axioms) are connected with structure. Provided if it is related with structures then it gives the idea of non-linearity beautifully. In other ways we can rightly say soft topology is somewhat directly proportional to non-linearity. Although we use non-linearity in Applied Math. So it is not wrong to say that soft topology is applied Math in itself. It means that soft topology has the taste of both of pure and applied math. In future I will discuss Separation Axioms in soft topology With respect to soft points. We expect that these results in this article will do help the researchers for strengthening the tool box of soft topological structures. In the forthcoming, we spread the idea of soft $\alpha$-open, and soft $b^*$ open sets in soft quad topological structure with respect to ordinary and soft points.

References


[38] S. Hussain, B. Ahmad, 2015, soft separation axioms in soft topological space submitted to elsvier.


