The $R_0$ and $R_1$ Properties in Fuzzy Soft Topological Spaces

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Abstract — The purpose of this paper is to introduce and study some new properties so-called fuzzy soft $R_i$ (for short, $FSR_i$, $i = 0, 1$) on fuzzy soft spaces by using quasi-coincident relation for fuzzy soft points, we get some characterizations and properties of them. Also, the relationships of these properties in fuzzy soft topologies which are constructed from crisp topology and soft topology over $X$ and vice versa are studied with some illustrative examples.

Keywords — Fuzzy soft set, Fuzzy soft point, Fuzzy soft quasi-coincident, Fuzzy soft topology.

1 Introduction

In 1999, Molodtsov [8] introduced the concept of soft set as one of mathematical tools for dealing with uncertainties. The works on the soft set theory have been applied in several directions. Maji et al.[7] introduced the concept of fuzzy soft set with some its properties. Then fuzzy soft theory and its applications have been studied by many authors. Chang [2] introduced the concept of fuzzy topology. Tanay et al.[12] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [9] gave the definition of fuzzy soft topology over the initial universe set. In recent time, many of notions and results in fuzzy soft topology have been studied as in [1, 3, 4, 5, 10].

In this paper, we define and study some new properties and results related to fuzzy soft spaces. The main aim of our work is to introduce and study the $R_0$ and $R_1$ properties in fuzzy soft topological spaces by using quasi-coincidence for fuzzy soft points. Some characterizations and basic properties of them are studied. Also we, investigate the relationships of these properties in fuzzy soft topologies which are derived from crisp topology and vice versa with some necessary examples.

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2 Definitions and Notions

Throughout this work, $X$ refers to a universe set, $E$ be the set of all parameters for $X$, $P(X)$ is the power set of $X$ and $I^X$ be the set of all fuzzy subsets of $X$, where $I = [0, 1]$.

**Definition 2.1.** [2, 13] A fuzzy set $A$ on $X$ is a set characterized by a membership function $\mu_A : X \rightarrow I$ whose value $\mu_A(x)$ represents the degree of membership of $x$ in $A$ for $x \in X$. A fuzzy point $x_\lambda$ $(0 < \lambda \leq 1)$ is a fuzzy set in $X$ given by $x_\lambda(y) = \lambda$ at $x = y$ and $x_\lambda(y) = 0$ otherwise for all $y \in X$. Here $x$ and $\lambda$ are called support and the value of $x_\lambda$, respectively. The set of all fuzzy point in $X$ denoted by $FP(X)$. For $\alpha \in I$, $\alpha \in I^X$ refers to the fuzzy constant function where, $\alpha(x) = \alpha \forall x \in X$ and for $x_\lambda \in FP(X)$, $O_{x_\lambda}$ refers to a fuzzy open set contains $x_\lambda$ and called fuzzy open neighborhood of $x_\lambda$. For $A$, $B \in I^X$, the basic operations for fuzzy sets are given by Zadah [13].

**Definition 2.2.** [6, 8] A soft set $F_E = (F, E)$ over $X$ with the set $E$ of parameters is a mapping $F : E \rightarrow P(X)$ the value $F(e)$ is a set called $e$-element of the soft set for all $e \in E$. Thus a soft set over $X$ can be represented by the set of ordered pairs $F_E = \{(e, F(e)) : e \in E, F(e) \subseteq P(X)\}$, we denote the family of all soft sets over $X$ by $SS(X, E)$.

**Definition 2.3.** [6, 11] Let $F_E \in SS(X, E)$ be a soft set over $X$. Then:

i. $F_E$ is called a null soft set, denoted by $\emptyset_E$, if $F(e) = \emptyset$ for every $e \in E$. And if $F(e) = X$ for all $e \in E$, then $F_E$ is called an universal soft set, denoted by $X_E$.

ii. If $F(e) = \{x\}$ and $F(e') = \emptyset$ for all $e' \in E - \{e\}$, then $F_E$ is called a soft point and denoted by $x^e$. The complement of a soft point $x^e$ is a soft set over $X$ denoted by $(x^e)'$ and given by $(x^e)'(e) = X - \{x\}$, $(x^e)'(e') = X$ for all $e' \in E - \{e\}$.

The set of all soft points over $X$ is denoted by $SP(X, E)$.

**Definition 2.4.** [7, 9] A fuzzy soft set $f_E = (f, E)$ over $X$ with the set $E$ of parameters is defined by the set of ordered pairs $f_E = \{(e, f(e)) : e \in E, f(e) \in I^X\}$. Here $f$ is a mapping given by $f : E \rightarrow I^X$ and the value $f(e)$ is a fuzzy set called $e$-element of the fuzzy soft set for all $e \in E$. The family of all fuzzy soft sets over $X$ is denoted by $FSS(X, E)$.

**Definition 2.5.** [7, 9] Let $f_E, g_E$ are two fuzzy soft sets over $X$. Then:

i. $f_E$ is called a null fuzzy soft set, denoted by $\hat{0}_E$ if $f(e) = 0$ for all $e \in E$. And if $f(e) = 1$ for all $e \in E$, then $f_E$ is called universal fuzzy soft, denoted by $\hat{1}_E$.

ii. A fuzzy soft $f_E$ is subset of $g_E$ if $f(e) \leq g(e)$ for all $e \in E$, denoted by $f \subseteq g$.

iii. $f_E$ and $g_E$ are equal if $f_E \subseteq g_E$ and $g_E \subseteq f_E$. It is denoted by $f_E = g_E$.

iv. The complement of $f_E$ is denoted by $f_E^c$, where $f^c : E \rightarrow I^X$ is a mapping defined by $f(e)^c = 1 - f(e)$ for all $e \in E$. Clearly, $(f_E^c)^c = f_E$.

v. The union of $f_E, g_E$ is a fuzzy soft set $h_E$ defined by $h(e) = f(e) \cup g(e)$ for all $e \in E$. $h_E$ is denoted by $f_E \cup g_E$. 
vi. The intersection of \( f_E \) and \( g_E \) is a fuzzy soft set \( l_E \) defined by, \( l(e) = f(e) \cap g(e) \) for all \( e \in E \). \( l_E \) is denoted by \( f_E \cap g_E \).

**Definition 2.6.** [1] A fuzzy soft point \( x_\alpha^e \) over \( X \) is a fuzzy soft set over \( X \) defined as follows:
\[
x_\alpha^e(e') = \begin{cases} 
\alpha & \text{if } e' = e \\
0 & \text{if } e' \in E \setminus \{e\}
\end{cases}
\]
where, \( x_\alpha \) is the fuzzy point in \( X \) with support \( x \) and value \( \alpha, \alpha \in (0,1] \). The set of all fuzzy soft points in \( X \) is denoted by \( FSP(X,E) \). The fuzzy soft point \( x_\alpha^e \) is called belongs to a fuzzy soft set \( f_E \), denoted by \( x_\alpha^e \in f_E \) iff \( \alpha \leq f(e)(x) \). Every non-null fuzzy soft set \( f_E \) can be expressed as the union of all the fuzzy soft points belonging to \( f_E \). The complement of a fuzzy soft point \( x_\alpha^e \) is a fuzzy soft set over \( X \).

**Definition 2.7.** [1, 9] Let \( X \) be a universe set, \( E \) be a fixed set of parameters and \( \delta \) be the family of fuzzy soft sets over \( X \), then \( \delta \) is said to be a fuzzy soft topology on \( X \) iff:

i. \( \hat{0}_E, \hat{1}_E \) belong to \( \delta \),

ii. The union of any number of fuzzy soft sets in \( \delta \) is in \( \delta \),

iii. The intersection of any two fuzzy soft sets in \( \delta \) is in \( \delta \).

In this case, \((X,\delta,E)\) is called a fuzzy soft topological space. The members of \( \delta \) are called fuzzy soft open sets in \( X \), denoted by \( FSO(X,\delta,E) \). A fuzzy soft set \( f_E \) over \( X \) is called fuzzy soft closed in \( X \) iff \( f_E^C \in \delta \), the set of all fuzzy soft closed sets over \( X \), denoted by \( FSC(X,\delta,E) \).

**Notation.**[10] Let \((X,\delta,E)\) be a fuzzy soft topological space. For \( x_\alpha^e \in FSP(X,E) \) the fuzzy soft set \( O_{x_\alpha^e} \) refers to a fuzzy soft open set contains \( x_\alpha^e \) and \( O_{x_\alpha^e} \) is called a fuzzy soft open neighborhood of \( x_\alpha^e \). The fuzzy soft open neighborhood system of \( x_\alpha^e \) denoted by, \( N_E(x_\alpha^e) \) is the family of all its fuzzy soft open neighborhoods.

In general for, \( f_E \in FSS(X,E) \) the notation \( O_{f_E} \) refers to a fuzzy soft open set contains \( f_E \) and is called a fuzzy soft open neighborhood of \( f_E \).

**Definition 2.8.** [1, 9] Let \((X,\delta,E)\) be a fuzzy soft topological space and \( f_E \in FSS(X,E) \). Then:

i. The fuzzy soft interior of \( f_E \) is the fuzzy soft set denoted by \( f_E^o \) and given by \( f_E^o = \cup\{g_E : g_E \in \delta \text{ and } g_E \subseteq f_E\} \), that is \( f_E^o \) is a fuzzy soft open set. Indeed it is the largest fuzzy soft open set contained in \( f_E \).

ii. The fuzzy soft closure of \( f_E \) is the fuzzy soft set denoted by \( f_E^c \) and given by \( f_E^c = \cap\{g_E : g_E \in \delta^c \text{ and } f_E \subseteq g_E\} \), that is \( f_E^c \) is a fuzzy soft closed set. Clearly, \( f_E^c \) is the smallest fuzzy soft closed set over \( X \) which contains \( f_E \).

**Definition 2.9.** [4] Let \((X,\delta,E)\) be a fuzzy soft topological space and \( Y \subseteq X \). Let \( h_Y^E \) be a fuzzy soft set over \((Y,E)\) such that \( h_Y^E : E \rightarrow I^Y \) such that \( h_Y^E(e) \in I^Y \),

\[
h_Y^E(e)(x) = \begin{cases} 
1 & \text{if } x \in Y \\
0 & \text{if } x \notin Y
\end{cases}
\]

Let \( \delta_Y = \{h_Y^E \cap g_E : g_E \in \delta\} \), then the fuzzy soft topology \( \delta_Y \) on \((Y,E)\) is called
fuzzy soft subspace topology for \((Y, E)\) and \((Y, \delta_Y, E)\) is called a fuzzy soft subspace of \((X, \delta, E)\). If \(h^\alpha_E \in \delta\) (resp. \(h^\alpha_E \in \delta^c\)), then \((Y, \delta_Y, E)\) is called fuzzy open (resp. closed) soft subspace of \((X, \delta, E)\).

**Definition 2.10.** [10] For \(A \subseteq X\). The soft characteristic of \(A\), denoted by \(\vec{\chi}_A\) is a fuzzy soft set \(\vec{\chi}_A : E \to \mathcal{P}^X\) defined by, \(\vec{\chi}_A(e) = \chi_A \forall e \in E\), where \(\chi_A\) is the characteristic of \(A\), i.e. \(\vec{\chi}_A = \{(e, \chi_A) : e \in E\}\), where \(\chi_A : X \to \{0, 1\}\).

**Definition 2.11.** [10] Let \(f_E \in FSS(X, E)\). Then the soft support of \(f_E\), denoted by \(SSup(f_E)\) is a soft set given by, \(SSup(f_E) = \{(e, S(f(e)) : e \in E\}\), where \(S(f(e))\) is the support of fuzzy set \(f(e)\), which is given by the set \(S(f(e)) = \{x \in X : f(e)(x) > 0\}\) \(\subseteq X\).

**Definition 2.12.** [1] The fuzzy soft sets \(f_E\) and \(g_E\) in \((X, E)\) are called fuzzy soft quasi-coincident, denoted by \(f_Eqg_E\) iff there exist \(e \in E\), \(x \in X\) such that \(f(e)(x) + g(e)(x) > 1\). If \(f_E\) is not fuzzy soft quasi-coincident with \(g_E\), then we write \(f_E\tilde{q}g_E\), that is \(f_E\tilde{q}g_E\) iff \(f(e)(x) + g(e)(x) \leq 1\), i.e. \(f(e)(x) \leq g^c(e)(x)\) for all \(x \in X\) and \(e \in E\).

A fuzzy soft point \(x^c_{\alpha}\) is said to be soft quasi-coincident with \(f_E\), denoted by \(x^c_{\alpha}qf_E\) iff there exists \(e \in E\) such that \(\alpha + f_E(e)(x) > 1\).

**Proposition 2.13.** [1, 10] Let \(x^c_{\alpha}, y^c_{\beta} \in FSP(X, E), f_E, g_E, h_E \in FSS(X, E)\) and \(\{f_i_E : i \in I\} \subseteq FSS(X, E)\). Then we have:

1. \(f_E\tilde{q}g_E \iff f_E \subseteq g_E\),
2. \(f_E \cap g_E = \tilde{0}_E \implies f_E\tilde{q}g_E\),
3. \(f_E\tilde{q}g_E, h_E \subseteq g_E \implies f_E\tilde{q}h_E\),
4. \(f_E\tilde{q}g_E \iff x^c_{\alpha}qg_E, for \text{some } x^c_{\alpha} \tilde{e} f_E\),
5. \(x^c_{\alpha}qf_E \iff x^c_{\alpha} \tilde{e} f_E\),
6. \(f_E \subseteq g_E \iff (x^c_{\alpha}qf_E \implies x^c_{\alpha}qg_E \text{ for all } x^c_{\alpha})\),
7. \(f_E\tilde{q}f_E^c\),
8. If \(x^c_{\alpha}q(\cap_{i \in J} f_i E)\), then \(x^c_{\alpha}qf_i E\) for all \(i \in J\),
9. \(x \neq y \implies x^c_{\alpha}qy^c_{\beta}\) for all \(\alpha, \beta \in I\),
10. \(x^c_{\alpha}qy^c_{\beta} \iff x \neq y \text{ or } (x = y \text{ and } \alpha + \beta \leq 1)\).

**Lemma 2.14.** [10] Let \((X, \delta, E)\) be a fuzzy soft topological space and \(x^c_{\alpha} \in FSP(X, E)\). Then:

i. \(g_Eqf_E \iff g_E\tilde{q}f_E\) for all \(g_E \in FSO(X, \delta, E)\),
ii. \(x^c_{\alpha}qf_E \iff O_{x^c_{\alpha}}qf_E\) for all \(O_{x^c_{\alpha}} \in N_E(x^c_{\alpha})\).
Theorem 2.15. [10]

i. Let $(X, \tau)$ be a crisp topological space, then the family $\delta_{\tau} = \{ \tilde{\chi}_A : A \in \tau \}$ forms a fuzzy soft topology on $X$ induced by $\tau$.

ii. Every fuzzy soft topological space $(X, \delta, E)$ defines a crisp topology on $X$ in the form $\tau_{\delta} = \{ A \subseteq X : \tilde{\chi}_A \in \delta \}$ which is induced by $\delta$.

Theorem 2.16. [10]

i. Let $(X, \tau^*, E)$ be a soft topological space, then the collection $\delta_{\tau^*} = \{ f_E \in FSS(X, E) : Ssup(f_E) \in \tau^* \}$ defines the fuzzy soft topology on $X$ which is induced by $\tau^*$.

ii. Let $(X, \delta, E)$ be a fuzzy soft topological space, then the family $\tau^*_\delta = \{ Ssup(f_E) : f_E \in \delta \}$ defines the soft topology on $X$ which is induced by $\delta$.

Proposition 2.17. [10] Let $(X, \tau)$ be a topological space, $(X, \tau^*, E)$ be a soft topological space and $(X, \delta, E)$ be a fuzzy soft topological space. Then:

i. $\alpha E \in \delta_{\tau^*}$ for all $\alpha \in I$,

ii. $F_E \in \tau^* \implies \tilde{\chi}_{FE} \in \delta_{\tau^*}$, in particular $\delta_\Delta \subseteq \delta_{\tau^*}$.

3 Fuzzy Soft $R_i$-Spaces, $i = 0, 1$.

Definition 3.1. A fuzzy soft topological space $(X, \delta, E)$ is said to be:

i. Fuzzy soft $R_0$ (FSR$_0$, for short) iff for every $x^e_{\alpha}, y^e_{\beta} \in FSP(X, E)$ with $x^e_{\alpha} \overline{q} y^e_{\beta}$ implies $\overline{x^e_{\alpha} q y^e_{\beta}}$.

ii. Fuzzy soft $R_1$ (FSR$_1$, for short) iff for every $x^e_{\alpha}, y^e_{\beta} \in FSP(X, E)$ with $x^e_{\alpha} \overline{q} y^e_{\beta}$ implies there exist $O_{x^e_{\alpha}}$ and $O_{y^e_{\beta}} \in \delta$ such that $O_{x^e_{\alpha}} \overline{q} O_{y^e_{\beta}}$.

In the following we get some characteristics of $FSR_i$-spaces, $i = 0, 1$.

Theorem 3.2. Let $(X, \delta, E)$ be a fuzzy soft topological space. Then the following items are equivalent:

i. $(X, \delta, E)$ is $FSR_0$.

ii. $\overline{x^e_{\alpha}} \subseteq O_{x^e_{\alpha}}$ for all $O_{x^e_{\alpha}} \in \delta$.

iii. $\overline{x^e_{\alpha}} \subseteq \cap \{ O_{x^e_{\alpha}} : O_{x^e_{\alpha}} \in \delta \}$ for all $x^e_{\alpha} \in FSP(X, E)$.

Proof. i $\implies$ ii) Let $(X, \delta, E)$ be $FSR_0$ and $y^e_{\beta} \overline{q} x^e_{\alpha}$, then $x^e_{\alpha} \overline{q} y^e_{\beta}$ implies $y^e_{\beta} q O_{x^e_{\alpha}} \forall O_{x^e_{\alpha}}$. Hence $\overline{x^e_{\alpha}} \subseteq O_{x^e_{\alpha}} \forall O_{x^e_{\alpha}}$ (by 6) of Proposition 2.13).

ii $\implies$ iii) Obvious.

iii $\implies$ i) Let $\overline{x^e_{\alpha}} \subseteq \cap \{ O_{x^e_{\alpha}} : O_{x^e_{\alpha}} \in N_E(x^e_{\alpha}) \} \subseteq O_{x^e_{\alpha}} \forall O_{x^e_{\alpha}}$. Now let $x^e_{\alpha}, y^e_{\beta} \in FSP(X, E)$ with $x^e_{\alpha} \overline{q} y^e_{\beta}$, then $x^e_{\alpha} \in y^e_{\beta}^c = O_{x^e_{\alpha}}$ and so, by hypothesis $\overline{x^e_{\alpha}} \subseteq O_{x^e_{\alpha}} = \overline{y^e_{\beta^c}} = (y^e_{\beta})^c \subseteq (y^e_{\beta})^c \implies \overline{x^e_{\alpha} q y^e_{\beta}}$. Hence $(X, \delta, E)$ is $FSR_0$. 


Theorem 3.3. Let \((X, \delta, E)\) be a fuzzy soft topological space and \(f_E \in FSC(X, \delta, E)\). Then the following items are equivalent:

i. \((X, \delta, E)\) is \(FSR_0\).

ii. \(x^e_\alpha \bar{q} f_E\) implies there exists \(O_{f_E} \in \delta\) contains \(f_E\) such that \(x^e_\alpha \bar{q} O_{f_E}\).

iii. \(x^e_\alpha \bar{q} f_E \implies \overline{x^e_\alpha q f_E}\).

iv. \(x^e_\alpha \bar{q} y_\beta \implies \overline{x^e_\alpha q y_\beta}\).

Proof. i \implies ii) Let \((X, \delta, E)\) be \(FSR_0\), \(f_E \in FSC(X, \delta, E)\) and \(x^e_\alpha \bar{q} f_E\), then \(x^e_\alpha \in f^c_E = O_{x^e_\alpha} \implies \overline{x^e_\alpha} \subseteq f^c_E = O_{x^e_\alpha}\) (by Theorem 3.2) \implies f_E \subseteq \overline{x^e_\alpha} = O_{f_E}.\) Since \(x^e_\alpha \subseteq \overline{x^e_\alpha}\), then \(\overline{x^e_\alpha} \subseteq (x^e_\alpha)^\epsilon\). Hence \(x^e_\alpha \bar{q} x^e_\alpha = O_{f_E}\).

ii \implies iii) Let \(x^e_\alpha \bar{q} f_E\), then by hypothesis there exists \(O_{f_E}\) such that \(x^e_\alpha \bar{q} O_{f_E} \implies \overline{x^e_\alpha q f_E}\) (by ii. of Lemma 2.14).

iii \implies iv) it is clear.

evidence iv \implies i) Let \(x^e_\alpha\), \(y_\beta \in FSP(X, E)\) with \(x^e_\alpha \bar{q} y_\beta \implies \overline{x^e_\alpha q y_\beta}\) (by given). Since \(y_\beta \subseteq \overline{y_\beta}\), then \(\overline{x^e_\alpha q y_\beta}\). Hence \((X, \delta, E)\) is \(FSR_0\).

Theorem 3.4. Every \(FSR_1 - \text{space} \) is a \(FSR_0 - \text{space}\).

Proof. Obvious.

Corollary 3.5. Let \((X, \delta, E)\) be a fuzzy soft topological space. Then \((X, \delta, E)\) is \(FSR_1\) if and only if for all \(x^e_\alpha\), \(y^e_\beta \in FSP(X, E)\) with \(x^e_\alpha \bar{q} y_\beta \) implies there exist \(O_{x^e_\alpha}\), \(O_{y^e_\beta} \in \delta\) such that \(O_{x^e_\alpha} \bar{q} O_{y^e_\beta}\).

Proof. Follows from the above theorem and from ii. of Theorem 3.2.

Theorem 3.6. Every subspace \((Y, \delta_Y, E)\) of a \(FSR_1 - \text{space} \) \((X, \delta, E)\) is a \(FSR_1 - \text{space} \), \(i = 0, 1\).

Proof. As a sample we prove the case \(i = 1\).

Let \(x^e_\alpha\), \(y^e_\beta\) are fuzzy soft points in \((Y, E)\) with \(x^e_\alpha \bar{q} y_\beta\). Then \(x^e_\alpha\), \(y^e_\beta\) also in \((X, E)\) with \(x^e_\alpha \bar{q} y_\beta\). Since \((X, \delta, E)\) is \(FSR_1\), then there exist \(O_{x^e_\alpha}\), \(O_{y^e_\beta} \in \delta\) such that \(O_{x^e_\alpha} \bar{q} O_{y^e_\beta}\) and so, there exist \(O_{x^e_\alpha} = O_{y^e_\alpha} \cap h^Y_Y \subseteq \delta_Y\), \(O_{y^e_\beta} = O_{y^e_\beta} \cap h^Y_Y \subseteq \delta_Y\) such that \(O_{x^e_\alpha} \bar{q} O_{y^e_\beta}\). Hence \((Y, \delta_Y, E)\) is \(FSR_1\)

Lemma 3.7. Let \((X, \tau)\) and \((X, \tau^e, E)\) be a topological space and a soft topological space respectively, then we have:

i. \(\overline{\chi_{\{x\}}} = \overline{\chi_{\{x\}}^\tau}\) for all \(x \in X\).

ii. \(\overline{\chi_{\{x\}}} = \overline{\chi_{\{x\}}^\tau^e}\) for all \(x^e \in SP(X, E)\).

Proof. Straightforward.

In the following, we introduce some relationships for \(FSR_1\)-axioms, \(i = 0, 1\) in fuzzy soft topologies and that on crisp and soft topologies.

Theorem 3.8. Let \((X, \tau)\) be a topological space. Then \((X, \delta, E)\) is a \(FSR_1\)-space if and only if \((X, \tau)\) is an \(R_i - \text{space} \), \(i = 0, 1\).
Proof. 1) For the case $i = 0$. Let $(X, \delta, E)$ be $FSR_0$ and $x \in \overline{y}$. Then $x_i^c \overline{\emptyset}^{\emptyset}$ and $x_i^c \emptyset \overline{y}$ (by i. of the above lemma). Since $(X, \delta, E)$ is $FSR_0$, then $x_i^c \emptyset O_y = y_i^c \overline{x_i^c}$ (by ii. of Lemma 2.14). Thus $y_i^c \overline{x_i^c}$ and so, $y \in \overline{x}$ (by i. of the above lemma). Hence $(X, \tau)$ is an $R_0$–space.

Conversely, let $(X, \tau)$ be $R_0$ and $x_\alpha, y_\beta \in FSP(X, E)$ with $x_\alpha \emptyset y_\beta$, in particular $x_\alpha \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{x_\alpha} \implies x \in \overline{y}$ (by i. of the above lemma). Since $(X, \tau)$ is $R_0$, then $y \in \overline{x} \implies y_i^c \overline{x_i^c} = \overline{y_i^c} \implies y_\beta \emptyset \overline{x_i^c} = \overline{y_\beta}$ (by i. of the above lemma). Hence we obtain the result.

2) For the case $i = 1$. Let $(X, \tau)$ be $R_1$ and $x_\alpha, y_\beta \in FSP(X, E)$ with $x_\alpha \emptyset y_\beta$, in particular $x_\alpha \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{x_\alpha} \implies x \in \overline{y}$, then there exist $O_x, O_y \in \tau$ such that $O_x \cap O_y = \emptyset$. Take $O_{x_\alpha} = \overline{\chi_{O_x}} \in \delta$, and $O_{y_\beta} = \overline{\chi_{O_y}} \in \delta$, then $O_{x_\alpha} \emptyset O_{y_\beta}$. Hence $(X, \delta, E)$ is $FSR_1$.

Conversely, let $(X, \delta, E)$ be $FSR_1$ and $y \not\in \overline{y}$ then there exists $x \in X$ such that $x \in \overline{y}$ and $x \not\in \overline{y}$ implies $x \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{y_\beta} \implies x_\alpha \emptyset \overline{x_\alpha}$, then there exist $O_{x_\alpha}, O_{y_\beta} \in \delta$ such that $O_{x_\alpha} \emptyset O_{y_\alpha}$ and so, there exist $O_x, O_y \in \tau$ such that $O_{x_\alpha} = \overline{\chi_{O_x}}$ and $O_{y_\beta} = \overline{\chi_{O_y}}$, then $\overline{\chi_{O_x}} \subseteq \overline{\chi_{O_y}} \implies O_x \subseteq \overline{O_y} \implies O_x \cap O_y = \emptyset$. Hence the result holds.

Theorem 3.9. Let $(X, \delta, E)$ be a fuzzy soft topological space. If $(X, \delta, E)$ is a $FSR_0$–space, then $(X, \tau_\delta)$ is a $R_0$–space.

Proof. It is similar to that of the necessity part of the above theorem.

Note. An $R_i$–space $(X, \tau_\delta)$ need not imply $(X, \delta, E)$ $FSR_i$–space, $i = 0, 1$, this fact can be shown by the following examples.

Examples 3.10. 1) Let $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$, then the family $\delta = \{0_x, 1_E, f_E = \{(e_1, (x, y, 0.5)), (e_2, 1)\})$ is a fuzzy soft topology on $X$ and $\delta = \{0, 1\}$ is a topology on $X$ which is induced by $\delta$. It is easy to check that $(X, \delta)$ is $R_0$, but the fuzzy soft topological space $(X, \delta, E)$ is not $FSR_0$. Indeed, for $x_{0.5}^c \in FSP(X, E)$, $x_{0.5}^c = \{(e_1, (x_{0.5}, y_1, z_1)), (e_2, 1)\}$, but there exists $O_{x_{0.5}^c} = \{(e_1, (x_{0.5}, y_1)), (e_2, 1)\}$ such that $x_{0.5}^c \not\subseteq O_{x_{0.5}^c}$.

2) Let $X = \{a, b, c\}$ and $E = \{e_1, e_2\}$. Then the family $\delta = \{0_a, 1_E, f_E\}$, where $f_E = \{(e_1, (a_{0.3}, c_{0.5})), (e_2, (a_{0.3}, c_{0.5}))\}$ is a fuzzy soft topology on $X$ and $\tau_\delta = \{0, 1\}$ is a topology on $X$ which is induced by $\delta$. It is clear that $(X, \tau_\delta)$ is $R_1$, but $(X, \delta, E)$ is not $FSR_1$, because for $a_{0.3}^c, b_{1.2}^c \in FSP(X, E)$ with $a_{0.3}^c \not\subseteq b_{1.2}^c$ implies there exist two open sets $F_E, G_E$ contains $x^c$ and $y^c$ respectively, such that $F_E \cap G_E = \emptyset$.

Definition 3.11. A soft topological space $(X, \tau^*, E)$ is said to be:

i. Soft $R_0$( for short, $SR_0$) iff for every pair of soft points $x^c$, $y^c (x \neq y) \in SP(X, E)$ with $x^c \subseteq \overline{y^c}$ implies $y^c \subseteq \overline{x^c}$.

ii. Soft $R_1$( for short, $SR_1$) iff for every pair of soft points $x^c$, $y^c (x \neq y) \in SP(X, E)$ with $x^c \not\subseteq \overline{y^c}$ implies there exist two open sets $F_E, G_E$ contains $x^c$ and $y^c$ respectively, such that $F_E \cap G_E = \emptyset$.

Theorem 3.12. Let $(X, \tau^*, E)$ be a soft topological space, then we have:
i. \((X, \delta^*, E)\) is FSR0 if and only if \((X, \tau^*, E)\) is SR0.

ii. If \((X, \tau^*, E)\) is SR1, then \((X, \delta^*, E)\) is FSR1.

**Proof.** i.) Let \((X, \delta^*, E)\) be FSR0 and \(x^e \in \overline{y^e}\), then \(x^e_i \bar{q} y^e_i\) (by Lemma 3.7). Since \((X, \delta^*, E)\) is FSR0, then \(x^e_i \bar{q} O_{y^e_i} \Rightarrow y^e_i \bar{q} x^e_i\) (by ii. of Lemma 2.14) \(\Rightarrow y^e_i \not\in \overline{x^e_i}\) (by Lemma 3.7). Hence \((X, \tau^*, E)\) is SR0.

Conversely, let \((X, \tau^*, E)\) be SR0 and \(x^e_\alpha \in FSP(X, E)\). Since \(x^e_\alpha \in \delta^* \forall \alpha \in I - \{0, 1\}\), then \(\overline{x^e_\alpha} = x^e_\alpha \cap O_{x^e_\alpha} \forall O_{x^e_\alpha}\). When \(\alpha = 1\), then clearly \(\overline{x^e_1} = O_{x^e_1}\). Hence we obtain the result.

ii.) Let \((X, \tau^*, E)\) be SR1 and \(x^e_\alpha, y^e_\beta \in FSP(X, E)\) with \(x^e_\alpha \bar{q} y^e_\beta \Rightarrow x^e_\alpha \bar{q} \overline{y^e_\beta}\). Then we have, either \(x \neq y\) or \(x = y\) and \(\alpha + \beta \leq 1\) (by 10. of Proposition 3.13).

Case I. If \(x \neq y\), then \(x^e \neq \overline{y^e} \Rightarrow (\overline{x^e} \neq \overline{y^e} \text{ or } \overline{x^e} = \overline{y^e})\). Now we have:

a. If \(\overline{x^e} \neq \overline{y^e}\), then there exist \(O_{\alpha^*}, O_{\beta^*} \in \tau^*\) such that \(O_{\alpha^*} \bar{q} O_{\beta^*} = \emptyset_E\). Take \(O_{x^e_\alpha} = \overline{\alpha_{x^e_\alpha}} \in \delta^*\) and \(O_{y^e_\beta} = \overline{\beta_{y^e_\beta}} \in \delta^*\), then \(O_{x^e_\alpha} \bar{q} O_{y^e_\beta}\). Hence \((X, \delta^*, E)\) is a FSR1-space.

b. If \(\overline{x^e} = \overline{y^e}\), then this case is excluded (since \((X, \tau^*, E)\) is SR1).

Case II. If \((x = y\) and \(\alpha + \beta \leq 1\)). Take \(O_{x^e_\alpha} = \overline{\alpha_E} \in \delta^*\), \(O_{y^e_\beta} = \overline{\beta_E} \in \delta^*\), then \(O_{x^e_\alpha} = \overline{\alpha_E \bar{q}} O_{y^e_\beta} = \overline{\beta_E}\). Hence \((X, \delta^*, E)\) is a FSR1-space.

**Note.** A soft \(R_i\)-space \((X, \tau^*_\delta, E)\) need not imply \((X, \delta, E)\) FSR\(i\), \(i = 0, 1\), this fact can be shown by the following example.

**Example 3.13.** Let \(X = \{a, b\}\) and \(E = \{e_1, e_2\}\). The family \(\delta = \{0_E, \bar{1}_E, f_E, g_E, h_E\}\), where \(f_E = \{(e_1, a, 0.6), (e_2, a, 0.6)\}, g_E = \{(e_1, b, 0.9), (e_2, b, 0.9)\}, h_E = \{(e_1, (a, 0.6), b, 0.9)), (e_2, (a, 0.6), b, 0.9)\}\) is a fuzzy soft topology on \(X\) and \(\tau^*_\delta = \{0_E, X_E, E = \{(e_1, a), (e_2, a)\}\}, G_E = \{(e_1, (b), (e_2, (b))\}\}\) is a soft topology on \(X\) which is induced by \(\delta\). It is clear that \((X, \tau^*_\delta, E)\) is soft \(R_1\), but \((X, \delta, E)\) is not FSR0, because for \(a^e_0, b^e_1 \in FSP(X, E)\) with \(a^e_0 \bar{q} b^e_1\), but \(b^e_1 \not\in \overline{a^e_0}\). Also, it is not FSR1, because for \(a^e_0, b^e_1 \in FSP(X, E)\) with \(a^e_0 \bar{q} b^e_1\) \(\Rightarrow O_{a^e_0} \bar{q} O_{b^e_1}\) for all \(O_{a^e_0}, O_{b^e_1} \in \delta\).

4 Conclusion

In this paper, we defined and studied some new axioms are called the \(R_0\) and \(R_1\) properties in fuzzy soft topological spaces and some of its properties. Also, the relationships of these properties are studied. We hope these basic results will help the researchers to enhance and promote the research on fuzzy soft theory and its applications. In the next work, by the same manner, we defined and study a new set of separation axioms on fuzzy soft spaces.

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References


